Spring 2023

Applied Algebra Qualifying Exam: Part A

1:00pm–4:00pm (PDT)
Tuesday May 23rd

- Write your name and student PID at the top right corner of each page of your submission.
- Do all four problems. Show your work.
- This part of the exam will represent 40% of the total score.
- Notation:
  - $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex components.
  - $\mathcal{M}_n$ denotes the set $\mathcal{M}_{m,n}$ with $m = n$.
  - $\mathbb{C}^n$ is the set of column vectors with n complex components.
  - x^{H} is the Hermitian transpose of a vector or matrix x.

Question 1.

(a) State, *but do not prove*, the Schur decomposition theorem.

(b) Let (λ, x) be a simple eigenpair of $A \in \mathcal{M}_n$ with $x^{\mathrm{H}}x = 1$.

(i) Prove that there exists a nonsingular matrix $\begin{pmatrix} x & X \end{pmatrix}$ with inverse $\begin{pmatrix} y & Y \end{pmatrix}^{\mathrm{H}}$ such that

$$\begin{pmatrix} y^{\mathrm{H}} \\ Y^{\mathrm{H}} \end{pmatrix} A \begin{pmatrix} x & X \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}, \quad \text{with } M \in \mathcal{M}_{n-1}.$$

(ii) Hence prove that the angle θ between x and y satisfies $\sec\theta = \|y\|_2$.

Question 2. Consider a Hermitian matrix $A \in \mathcal{M}_n$.

(a) Show that the eigenvalues of A are real.

(b) Assume that the eigenvalues of A are ordered so that $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_2 \le \lambda_1$. State, *but do not prove*, the Courant-Fischer theorem.

(c) Prove that

$$\lambda_n = \min_{x^{\mathrm{H}}x=1} x^{\mathrm{H}}Ax, \quad \text{and} \quad \lambda_1 = \max_{x^{\mathrm{H}}x=1} x^{\mathrm{H}}Ax.$$

Question 3.

(a) Given a vector norm $\|\cdot\|$, define the matrix norm subordinate to $\|\cdot\|$. Prove that every subordinate matrix norm is consistent.

(b) Given $A \in \mathcal{M}_{m,n}$, let $\|A\|$ denote the matrix norm subordinate to the vector norm $\|\cdot\|$. Prove that $\|xy^{\mathrm{H}}\| = \|x\|\|y\|_D$ for all x, $y \in \mathbb{C}^n$, where $\|y\|_D$ denotes the vector norm dual to $\|\cdot\|$.

(c) Given $A \in \mathcal{M}_{m,n}$, prove that $\|Ax\|_2 \le \|A\|_F\|x\|_2$ for all $x \in \mathbb{C}^n$.

(d) Find $\|xy^{\mathrm{H}}\|$ for the Frobenius norm.

Question 4.

(a) State, *but do not prove*, the singular-value decomposition theorem.

(b) For a given $A \in \mathcal{M}_{m,n}$, prove that

$$\sigma_1(A) = \max_{x,y\neq 0} \frac{|y^{\mathrm{H}}Ax|}{\|y\|_2\|x\|_2},$$

where $\sigma_1(A)$ is the largest singular value of A.

(c) For any $A \in \mathcal{M}_n$, define (i) the field of values $\mathcal{F}(A)$; (ii) the spectral radius $\rho(A)$; and (iii) the numerical radius $\omega(A)$. Prove that $\rho(A) \le \omega(A) \le \sigma_1(A)$.