Instructions: Do all problems. All problems are weighted equally. You are not allowed to consult any external resource during this exam. Good luck!

Problem 1: Let $D_6$ be the group of symmetries of a regular hexagon, let $S_3$ be the symmetric group on three objects, and let $C_2$ be the cyclic group of order 2.

(1) Prove that $D_6$ is isomorphic to the direct product $S_3 \times C_2$.

(2) Calculate the character table of $D_6$. 
Problem 2: Let $X$ be the 9-element set of positions in a $3 \times 3$ matrix. The dihedral group $D_4$ of symmetries of a square acts on $X$ in a natural way. Let $\mathbb{C}[X]$ be the corresponding permutation representation and let $R : \mathbb{C}[X] \to \mathbb{C}[X]$ be the operator defined by

$$R(v) := \frac{1}{|D_4|} \sum_{g \in D_4} g \cdot v$$

for all $v \in \mathbb{C}[X]$. What is the rank of the linear operator $R$?
Problem 3: The symmetric group $S_5$ acts on the set $X$ of ordered pairs $(i, j)$ of (not necessarily distinct!) elements of \{1, 2, 3, 4, 5\}. Let $\mathbb{C}[X]$ be the associated permutation representation.

1) Find the decomposition of $\mathbb{C}[X]$ into irreducible $S_5$-modules.
2) Find the dimension of the endomorphism algebra $\text{End}_{S_5}(\mathbb{C}[X])$. 
**Problem 4:** Let $G = \mathbb{Z}$ be the group of integers under addition. The group homomorphism $\rho : G \to GL_2(\mathbb{C})$ given by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

gives $\mathbb{C}^2$ the structure of a $G$-module. Does $\mathbb{C}^2$ admit a $G$-invariant inner product?