Applied Algebra Qualifying Exam: Part B
Spring 2023

Instructions: Do all problems. All problems are weighted equally. You are not allowed to consult any external resource during this exam. Good luck!

Problem 1: Let D_6 be the group of symmetries of a regular hexagon, let S_3 be the symmetric group on three objects, and let C_2 be the cyclic group of order 2.

(1) Prove that D_6 is isomorphic to the direct product $S_3 \times C_2$.
(2) Calculate the character table of D_6.

Problem 2: Let X be the 9-element set of positions in a 3×3 matrix. The dihedral group D_4 of symmetries of a square acts on X in a natural way. Let $\mathbb{C}[X]$ be the corresponding permutation representation and let $R : \mathbb{C}[X] \to \mathbb{C}[X]$ be the operator defined by

$$R(v) := \frac{1}{|D_4|} \sum_{g \in D_4} g \cdot v$$

for all $v \in \mathbb{C}[X]$. What is the rank of the linear operator R?

Problem 3: The symmetric group S_5 acts on the set X of ordered pairs (i, j) of (not necessarily distinct!) elements of $\{1, 2, 3, 4, 5\}$. Let $\mathbb{C}[X]$ be the associated permutation representation.

(1) Find the decomposition of $\mathbb{C}[X]$ into irreducible S_5-modules.
(2) Find the dimension of the endomorphism algebra $\mathrm{End}_{S_5}(\mathbb{C}[X])$.

Problem 4: Let $G = \mathbb{Z}$ be the group of integers under addition. The group homomorphism $\rho : G \to GL_2(\mathbb{C})$ given by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

gives $\mathbb{C}^2$ the structure of a G-module. Does $\mathbb{C}^2$ admit a G-invariant inner product?