Spring 2026 Applied Algebra Qualifying Exam

Math 202 A/B/C

Name: ____

Instructions.

1. There are ten problems on this exam.
2. Give complete proofs and justify all assertions.
3. Throughout this exam, the term *Hilbert space* refers to a finite-dimensional complex vector space V equipped with a scalar product.
4. The term *algebra* refers to a finite-dimensional complex vector space $\mathcal{A}$ equipped with an associative, bilinear, unital multiplication and an antilinear, antimultiplicative, involutive conjugation.
5. All algebra homomorphisms respect multiplication, conjugation, and units.

Problem 1.

Let V and W be Hilbert spaces, each of dimension at least two. Prove that the operator norm on $\mathrm{Hom}(V, W)$ is not induced by any scalar product on $\mathrm{Hom}(V, W)$.

Problem 2.

Let A be an $m \times n$ matrix whose columns are one-hot binary vectors: each column contains exactly one entry equal to 1 and all other entries are equal to 0. Let r_i denote the sum of the entries in row i of A. Determine the nonzero singular values of A, and give explicit orthonormal bases for each of the corresponding left and right singular spaces.

Problem 3.

Let (V, φ) and (W, ψ) be finite-dimensional unitary representations of a group G. Let $A \in \mathrm{Hom}_G(V, W)$ be a G-equivariant linear map with operator norm $\sigma_1 > 0$. Prove that

$$V_1 = \{v \in V : \|Av\| = \sigma_1 \|v\|\}$$

is a subrepresentation of (V, φ).

Problem 4.

Let $\mathcal{A}$ be an algebra. Prove that $\mathcal{A}$ is commutative if and only if all its elements are normal.

Problem 5.

Let $\mathcal{A}$ be an algebra which admits a faithful state $\sigma\colon \mathcal{A} \to \mathbb{C}$. Prove that $\mathcal{A}$ admits a faithful tracial state $\tau\colon \mathcal{A} \to \mathbb{C}$.

Problem 6.

Let $\mathcal{B}$ be a commutative subalgebra of an algebra $\mathcal{A}$. Prove that $\mathcal{B}$ is a maximal commutative subalgebra of $\mathcal{A}$ if and only if it is equal to its own centralizer. Using this result, or otherwise, classify the maximal commutative subalgebras of $\mathrm{End}(V)$, where V is a Hilbert space.

Problem 7.

Let $G = (S, T)$ be the graph whose vertex set S consists of all bitstrings of length 3, with $\{s_1, s_2\} \in T$ an edge if and only if s_1 and s_2 differ by a single bit. Let $A \in \operatorname{End} \mathcal{F}(S)$ be the adjacency operator of G. Find the eigenvalues of A, and give an orthonormal basis for each eigenspace.

Problem 8.

Let G be a nontrivial finite group of odd order. Show that the convolution algebra $\mathcal{C}(G)$ contains a subalgebra $\mathcal{B}$ not isomorphic to $\mathcal{C}(H)$ for any subgroup $H \leq G$.

Problem 9.

Let $n \geq 2$. Let $S_n = \mathrm{Aut}\{1, \dots, n\}$ be the symmetric group, and consider the unitary representation (V, φ) of S_n in which

$$V = \bigoplus_{i=1}^{n} \mathbb{C}|i\rangle, \qquad \varphi(\pi)|i\rangle = |\pi(i)\rangle.$$

Decompose (V, φ) into irreducible unitary representations of S_n, with proof. Write out the character table of S_3.

Problem 10.

Let G be a finite group and let H be a subgroup of G. Let $\Lambda(G)$ be a set parameterizing irreducible unitary representations V^λ of G, and let $\Lambda(H)$ be a set parameterizing irreducible unitary representations W^μ of H. For $\lambda \in \Lambda(G)$ and $\mu \in \Lambda(H)$, let $m_{\lambda\mu}$ be the multiplicity of W^μ in the restriction of V^λ to H. Omitting terms with $m_{\lambda\mu} = 0$, show that the centralizer of $\mathcal{C}(H)$ in $\mathcal{C}(G)$ is isomorphic to

$$\bigoplus_{\lambda\in\Lambda(G)} \bigoplus_{\mu\in\Lambda(H)} \operatorname{End}\Big(\mathbb{C}^{m_{\lambda\mu}}\Big).$$