Fall 2023

Applied Algebra Qualifying Exam: Part A

1:00pm–4:00pm (PDT)
Thursday September 7th

- Write your name and student PID at the top right corner of each page of your submission.
- Do all four problems. Show your work.
- This part of the exam will represent 40% of the total score.
- Notation:
  - $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex components.
  - $\mathcal{M}_n$ denotes the set $\mathcal{M}_{m,n}$ with $m = n$.
  - $\mathbb{C}^n$ is the set of column vectors with n complex components.
  - x^{T} is the transpose of a vector or matrix x.
  - x^{H} is the Hermitian transpose of a vector or matrix x.
  - $\mathrm{eig}(A)$ is the set of eigenvalues of the matrix A (counting multiplicities).
  - $\mathrm{Re}(\alpha)$ is the real part of the complex scalar α.
  - $\mathrm{Im}(\alpha)$ is the imaginary part of the complex scalar α.

Question 1.

(a) State, *but do not prove*, the Schur decomposition theorem for a matrix $A \in M_n$.

(b) Prove that for $A, B \in \mathcal{M}_n$, if $x^{\mathrm{H}}Ax = x^{\mathrm{H}}Bx$ for all $x \in \mathbb{C}^n$, then $A = B$. Give an example for which $x^{\mathrm{T}}Ax = x^{\mathrm{T}}Bx$ for all $x \in \mathbb{C}^n$ but $A \neq B$.

Question 2.

(a) Prove that every $A \in \mathcal{M}_n$ may be written uniquely as $A = S + iT$, where S and T are Hermitian.

(b) For any $A \in \mathcal{M}_n$, consider the unique expansion $A = S + iT$, where S and T are Hermitian. Prove that for any $\lambda \in \operatorname{eig}(A)$, it holds that

$$\lambda_n(S) \leq \operatorname{Re}(\lambda) \leq \lambda_1(S) \quad \text{and} \quad \lambda_n(T) \leq \operatorname{Im}(\lambda) \leq \lambda_1(T),$$

where $\lambda_1(C)$ and $\lambda_n(C)$ denote the largest and smallest eigenvalues of a Hermitian matrix $C \in \mathcal{M}_n$.

Question 3. Consider any $A \in \mathcal{M}_{m,n}$.

(a) Define $|A|$, the modulus of A. Prove that the eigenvalues of $|A|$ are the singular values of A.

(b) Prove that if $m = n$, then $|A|$ and $|A^{\mathrm{H}}|$ are similar.

Question 4.

(a) Consider any Hermitian $A \in \mathcal{M}_n$ with eigenvalues ordered so that $\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A)$. Prove that

$$\lambda_n(A) \leq \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}x} \leq \lambda_1(A),$$

for all nonzero $x \in \mathbb{C}^n$.

(b) Suppose that $D \in \mathcal{M}_n$ with $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Prove that for all $1 \leq p \leq \infty$ the p-norm of D is given by $\|D\|_p = \max_{1 \leq i \leq n} |d_i|$.

(c) Given $a \in \mathbb{C}^n$, find $\|A\|_2$ for the matrices

$$A = aa^{\mathrm{H}} \quad \text{and} \quad A = \begin{pmatrix} 0 & a^{\mathrm{H}} \\ a & 0 \end{pmatrix}.$$

(Show your work. Simply writing down the answer will not be sufficient.)