Complex Analysis Qualifying Exam

1:00pm–4:00pm (PDT), in person
Thursday May 25, 2023

- Write your name and student PID at the top right corner of each page of your submission.
- The more detail you provide, the more accurately your work can be evaluated.
- By participating in this exam you are agreeing to abide by the UCSD Policy on Academic Integrity. The instructors reserve the right to require a follow-up oral examination.
- This is a closed-book examination. No cell-phone or Internet aids.
- Each problem is worth 10 points, except a bonus problem worth 3 points.
- Notation:
  - $\mathbb{C}$ denotes the complex plane.
  - $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$.
  - $\mathbb{T} = \partial\mathbb{D}$ denotes the unit circle in $\mathbb{C}$.
  - $\overline{\mathbb{D}}$ denotes the closed unit disc $\mathbb{D} \cup \mathbb{T}$.
  - $\mathcal{C}(\mathbb{K}) = \{\text{continuous } f \colon \mathbb{K} \to \mathbb{C}\}$.

Question 1. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence equal to one.

(a) Give an example where $A(z)$ diverges at each point of $\mathbb{T}$, and explain why.

(b) Give an example where $A(z)$ converges absolutely on $\mathbb{T}$, and explain why.

(c) Prove that $A(z)$ has a singularity on $\mathbb{T}$, i.e. cannot be analytically continued to an open set containing $\overline{\mathbb{D}}$.

(d) Prove that if $A(z)$ has nonnegative coefficients, the point $z = 1$ is a singularity of $A(z)$. (Hint: the previous problem could be useful.)

(e) Verify that $A(z) = \sum_{n=0}^{\infty} z^{n!}$ has radius of convergence one, and that each point of $\mathbb{T}$ is a singularity of $A(z)$.

Question 2. Let $z_1, \ldots, z_n$ be points on $\mathbb{T}$. Prove that that there is $z_0 \in \mathbb{T}$ such that the product of the distances from z_0 to each of the given points is at least one.

Question 3. By evaluating an appropriate contour integral, show that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2n}(t)dt = \frac{(2n-1)(2n-3)\dots(3)}{(2n)(2n-2)\dots(2)}.$$

Question 4. Let $f(z)$ be analytic on an open set containing $\overline{\mathbb{D}}$. Suppose that $|f(z)| > m$ for $|z| = 1$ and $|f(0)| < m$ for a positive number m. Prove that $f(z)$ has a zero in $\mathbb{D}$.

Question 5. Let $f(z)$ be analytic on $\mathbb{D}$. Prove the following statements.

(a) If $|f'(z) - f'(0)| < |f'(0)|$ for all $z \in \mathbb{D}$, then $f(z)$ is injective.

(b) If $|f(z)| < 1$ for all $z \in \mathbb{D}$ and $f(z)$ has two distinct fixed points, then $f(z) = z$. (Hint: Analytic automorphisms of $\mathbb{D}$ and Schwarz's Lemma could be useful.)

Question 6. Let $f(z) = \frac{\cos z}{z(z-5)}$.

(a) Prove that there is a sequence of rational functions $R_n(z)$ whose poles can only occur at 2 and 6 such that

$$\lim_{n\to\infty} \sup_{3\le|z|\le 4} |f(z) - R_n(z)| = 0. \tag{1}$$

(b) Does there exist a sequence of rational functions $R_n(z)$ whose poles can only occur at 6 such that (1) holds? Justify your answer.

Question 7. Let $a \in \mathbb{D}$ and $\mathbb{G} = \mathbb{D} \backslash \{a\}$.

(a) Construct a harmonic function v on $\mathbb{G}$ such that the following two conditions are satisfied:

$$\lim_{z \to z^*} v(z) = 0 \ \forall z^* \in \mathbb{T} \quad \text{and} \quad \lim_{z \to a} v(z) = +\infty.$$

Write down an explicit formula for the function v you constructed. (Hint: you only need to find one such function; first think about the case $a = 0$.)

(b) Let $f \in \mathcal{C}(\partial\mathbb{G})$ be defined as $f(z) = 0$ for $z \in \mathbb{T}$ and $f(a) = 2$. Prove that the Dirichlet problem on $\mathbb{G}$ with boundary data f has no solution: there is no harmonic function u on $\mathbb{G}$ such that $\lim_{z \to z_0} u(z) = f(z_0)$ for all $z_0 \in \partial\mathbb{G}$. (Hint: argue by contradiction, making use of the function you constructed in part (a).)

Question 8. This is an **optional bonus question**. Precisely formulate a problem/question in complex analysis whose solution/answer is not yet known to you, but which you are are eager to learn.