Complex Analysis Qualifying Exam

1:00pm–4:00pm (PDT), in person
Wednesday August 30, 2023

- Write your name and student PID at the top right corner of each page of your submission.
- By participating in this exam you are agreeing to abide by the UCSD Policy on Academic Integrity. The instructors reserve the right to require a follow-up oral examination.
- The more detail you provide, the more accurately your work can be evaluated.
- The exam is printed single-sided so that you do not need extra paper - you may write on each side of each page.
- This is a closed-book examination. No cell-phone or Internet aids.
- Each problem is worth 10 points.
- Notation:
  - $\mathbb{C}$ denotes the complex plane.
  - $\mathbb{R}$ denotes the real line in $\mathbb{C}$.
  - $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$.

Question 1. Let $G_N(z) = \sum_{n=0}^{N} z^n$ and $G(z) = (1-z)^{-1}$.

(a) Carefully prove that $G_N \to G$ as $N \to \infty$, uniformly on compact subsets of $\mathbb{D}$.

(b) What is the power series expansion of $G(z)$ at $z = 2023$, and what is its radius of convergence?

Question 2. Let $A_N(z) = \sum_{n=0}^{\infty} a_{Nn} z^n$ be a sequence of analytic functions on $\mathbb{D}$ which is uniformly bounded on compact subsets of $\mathbb{D}$. Let $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be an analytic function on $\mathbb{D}$ such that $\lim_{N\to\infty} a_{Nn} = b_n$ for each n.

(a) Prove that $A_N \to B$ as $N \to \infty$, uniformly on compact subsets of $\mathbb{D}$.

(b) Give an example showing that the conclusion in (a) may fail if uniform boundedness is dropped.

Question 3. Prove that the image of a nonconstant entire function $E\colon \mathbb{C} \to \mathbb{C}$ is dense in $\mathbb{C}$.

Question 4. Evaluate $\int_0^{2\pi} e^{e^{i\theta}}\,d\theta$, carefully explaining your solution.

Question 5. Let f be continuous on $\mathbb{C}$ and analytic on $\mathbb{C}\backslash\mathbb{R}$. Prove that f is analytic on $\mathbb{C}$.

Question 6. For each $N \in \mathbb{N}$, let $P_N(z) = \sum_{n=0}^{N} \frac{z^n}{n!}$.

(a) Show that the set $\{z \in \mathbb{C}\colon P_N(z) = 0 \text{ for some } N \in \mathbb{N}\}$ is discrete.

(b) Find a constant $c > 0$ such that $P_N(z)$ has no zeros in $\{z \in \mathbb{C} \colon |z| < cN\}$. You may use the inequality $n! > e^{-n}n^n$ if you wish.

Question 7. Determine whether the following statements are true or false. Justify your answer.

(a) Let $\mathbb{G} = \{z \in \mathbb{C} : |z| < 2 \text{ and } |z-1| > 1\}$. Then the set of polynomials is dense in the space $H(\mathbb{G})$ of analytic functions on $\mathbb{G}$.

(b) Let u_1, u_2 be harmonic functions on $\mathbb{D}$. Then $u := \max\{u_1, u_2\}$ is also harmonic on $\mathbb{D}$.

Question 8. Let $f(z)$ be a nowhere zero analytic function on the entire complex plane $\mathbb{C}$ and write $u(z) = \log|f(z)|$. Assume $|u|$ is Lebesgue integrable: $\int_{\mathbb{C}} |u(z)| dxdy < +\infty$, where $z = x + iy$. Prove f is constant. What is the value of f?