Complex Analysis Qualifying Exam – Spring 2022

Name: ________________________________

Student ID: ________________________________

Instructions: 3 hours. Open book: Conway and personal notes from lectures may be used. You may use without proof results proved in Conway I-VIII, X-XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: A region is an open and connected subset of \( \mathbb{C} \). The space of analytic (resp., meromorphic) functions in \( G \) is denoted by \( H(G) \) (resp., \( M(G) \)).

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Problem 1. [10 points.]

Let $G \subset \mathbb{C}$ be a bounded, simply connected region and let $a \in G$. Let $f$ be an analytic self-map of $G$ (i.e., $f(G) \subset G$) such that $f(a) = a$ and $f'(a) = 1$. Show that $f(z) = z$. 
Problem 2. [10 points; 4, 4, 2.] 

Let \( p(z) \) be a nonconstant polynomial of \( z \). Let \( G \subset \mathbb{C} \) be a component of the set \( \{ z : |p(z)| < 1 \} \).

(a) Show that \( p \) has at least one zero in \( G \).

(b) Let \( f \) be analytic in \( G \) with \( |f| \leq 1 \). Assume that \( f \) has a zero at every zero of \( p \) such that the order of vanishing of \( f \) is at least that of \( p \). Show that \( |f(z)| \leq |p(z)| \) and if \( z = a \) is a zero of \( p \) of order \( k \), then \( |f^{(k)}(a)| \leq |p^{(k)}(a)| \).

(c) If either \( |f(a)| = |p(a)| \) for some \( z = a \) that is not a zero of \( p \) or if \( |f^{(k)}(a)| = |p^{(k)}(a)| \) for some \( z = a \) that is a zero of \( p \) of order \( k \), then \( f(z) = cp(z) \) for some constant \( c \).
Problem 3. [10 points.]

Consider the function

$$f(z) = \frac{z^2 + 1}{z^2 - 1}$$

in $G = \{z : |z| > 2\}$. Does $f$ have a primitive in $G$ (i.e., $F \in H(G)$ such that $F' = f$)? Prove your assertion.
Problem 4. [10 points.]

Let $G \subset \mathbb{C}$ be a region such that $0 \not\in G$ and $G$ is not simply connected. Show that the following are equivalent:

(i) $\mathbb{C}_\infty \setminus G$ has precisely two components $F_0, F_\infty$ such that $0 \in F_0$, $\infty \in F_\infty$.

(ii) Every $f \in H(G)$ can be approximated in $H(G)$ by rational functions with poles only in $\{0, \infty\}$. 
Problem 5. [10 points.]

Let $G \subset \mathbb{C}$ be an open set, $\{f_n\}$ a sequence in $M(G)$, and $f$ a meromorphic function such that $f_n \rightarrow f$ in $M(G)$. Suppose $a \in G$ is a pole of $f$. Show that there is a sequence $\{a_n\}$ in $G$ such that $a_n \rightarrow a$ and $f_n$ has a pole at $a_n$ for sufficiently large $n$. 
Problem 6. [10 points.]

Let $h$ be a bounded harmonic function on the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$. Assume that

$$\limsup_{z \to a} h(z) \leq 0$$

for all $a \in \partial \mathbb{D} \setminus \{1\}$. Show that $h \leq 0$ in $\mathbb{D}$. 