Name ______________________________ **SID** ______________________________

Qualifying Exam in Numerical Analysis

Spring 2023

Time: 1:00 pm – 4:00 pm, Tuesday, May 16, 2023

	Full Scores	Your Scores
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Exam Rules and Instructions

- **This is a three-hour, close-book, and close-note exam. No calculators, computers, tablets, and any other electronic devices are allowed. No cheatsheets are allowed.**
- **There are a total of 12 pages (including this cover page) of the exam.**
- **You must show all the computational steps for how you get the answers. No credit will be given for unsupported answers.**
- **All numbers in computational results must be exact, whether written in rational/radical or decimal format. No credit will be given for rounded numbers.**

1. (25 points) Consider the following matrices and vectors

$$A = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 8 \end{bmatrix}.$$

Use the above to find the perturbation matrix δA such that $(A+\delta A)\hat{x} = b$ and $\|\delta A\|_2$ is the smallest. Give sufficient reasons to justify your answer.

2. (25 points) Consider the following matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & a & b \end{bmatrix}.$$

Use Cholesky factorization to determine the set of values for the pair (a, b) such that the matrix A is positive definite. Give sufficient reasons to justify your answer.

3. (25 points) Consider the iterative formula

$$x_{k+1} = \begin{bmatrix} \frac{3}{2} & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad k = 0, 1, 2, \ldots,$$

for solving a 3-by-3 linear system. Find the initial vectors x_0 such that the produced sequence $\{x_k\}_{k=1}^{\infty}$ converges and $\|x_0\|_2$ is the smallest. Give sufficient reasons to justify your answer.

4. (25 points)

(a) Use Newton's iteration to obtain the iterate (x_1, y_1), giving $(x_0, y_0) = (1, 0)$, for solving the system of equations $x + y - 2 = 0$ and $x \cos y + 1 = 0$. (Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.)

(b) Let $f \in C^1(\mathbb{R}^n)$, $x^{(0)} \in \mathbb{R}^n$, and $x^{(1)} = x^{(0)} - \alpha_0 \nabla f(x^{(0)})$, where

$$\alpha_0 = \arg\min\{f(x^{(0)} - \alpha \nabla f(x^{(0)})) : \alpha > 0\}.$$

(Assume the minimum is attained.) Show that $\nabla f(x^{(0)}) \cdot \nabla f(x^{(1)}) = 0$ and $f(x^{(1)}) \leq f(x^{(0)})$.

5. (25 points) Let $f \in C([a,b]) \setminus \mathcal{P}$ with $a, b \in \mathbb{R}$, $a < b$, and $\mathcal{P}$ the set of all real polynomials. Let $n \geq 0$ be an integer and $p \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$, where $\mathcal{P}_n$ is the set of all real polynomials of degree $\leq n$.

   (a) Prove that there exist $n+1$ distinct points $x_i \in [a,b]$ such that $p(x_i) = f(x_i)$ $(i = 1, \ldots, n+1)$.

   (b) Assume $f \in C^{n+1}([a,b])$ and $f^{(n+1)}(x) > 0$ for all $x \in [a,b]$. Prove that $f - p$ reaches it maximum in magnitude with alternative signs at exactly $n+2$ distinct points in $[a,b]$.

6. (25 points) Let P_n be the nth Legendre polynomial with $n \geq 2$. It is known that P_n' has $n-1$ distinct roots $x_1, \ldots, x_{n-1} \in (-1, 1)$. Denote $x_0 = -1$ and $x_n = 1$. Let $l_i \in \mathcal{P}_n$ $(i = 0, \ldots, n)$ be the basic Lagrange polynomials associated with x_i $(i = 0, \ldots, n)$, i.e.,

$$l_i(x) = \frac{\omega(x)}{\omega'(x_i)(x - x_i)} \quad (i = 0, 1, \ldots, n) \qquad \text{and} \qquad \omega(x) = a_{n-1} \prod_{i=0}^{n} (x - x_i) = (x^2 - 1) P_n'(x),$$

where a_{n-1} is the leading coefficient of P_n'. Consider the interpolatory quadrature

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=0}^{n} c_i f(x_i), \qquad c_i = \int_{-1}^{1} l_i(x)\, dx \quad (i = 0, 1, \ldots, n).$$

(a) Prove $c_0 = \frac{2}{n(n+1)}$ and $c_n = \frac{2}{n(n+1)}$. (You need only to prove one of them as the other is similar.)
(b) Prove that the degree of precision of the above quadrature is less than or equal to $2n - 1$.

Note. You can use the following without proof: $(1 - x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$, $P_n(1) = 1$, and $P_n(-1) = (-1)^n$.

7. (25 points) Consider the following method, for the initial value problem

$$y' = f(t, y),$$

with $y(t_0) = y_0$, that advances from $t = t_i$ to t_{i+1} through the steps:

- Form

  $$y_i^{(0)} = y_i,$$

  and

  $$y_i^{(1)} = y_i^{(0)} + hf(t_i, y_i^{(0)}),$$

  and

  $$y_i^{(2)} = \frac{3y_i^{(0)} + y_i^{(1)}}{4} + \frac{h}{4}f(t_i + h, y_i^{(1)}),$$

  where h is the stepsize $t_{i+1} - t_i$;

- Then form

  $$y_{i+1} = \frac{y_i^{(0)} + 2y_i^{(2)}}{3} + \frac{2h}{3}f(t_i + \frac{h}{2}, y_i^{(2)}).$$

Prove this is a Runge-Kutta method by finding its Butcher tableau. Remember to show your work and to simplify the values in the Butcher tableau.

8. (25 points) Consider the initial value problem consisting of

$$y' = f(t, y)$$

and $y(t_0) = y_0$, and consider the following individual solvers over evenly spaced nodes, spaced by stepsize h:

- Predictor:

$$y_{i+1} = y_{i-q} + h \sum_{j=0}^{\ell} \gamma_{\ell-j} f(t_{i-j}, y_{i-j}),$$

  for some constants $\gamma_{\ell-j}$, for $0 \leq j \leq \ell$, with local trunction error ρ_{i+1} of order s, where $s > 0$ is an integer;

- Corrector:

$$y_{i+1} = y_{i-p} + h \sum_{j=0}^{k} \beta_{k-j} f(t_{i+1-j}, y_{i+1-j}),$$

  for some constants β_{k-j}, for $0 \leq j \leq k$, with $\beta_k \neq 0$, with local truncation error σ_{i+1} of order r, where $r > 0$ is an integer.

Now consider a new method that advances a step using a prediction and two corrections, so the predictor is used to generate an initial guess that is then improved by two fixed-point iterations in the corrector. Find the order of the local truncation error of this new method in terms of r, s.