Ph.D./Masters Qualifying Examination
in Numerical Analysis

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10am–1pm
Wednesday May 30, 2007
5402 AP&M

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
1. **Norms, Condition Numbers, Linear Equations and Linear Least-Squares**

In Parts 1 and 2, $\| \cdot \|_p$ refers to the vector $p$-norm or its subordinate matrix norm.

**Question 1.1.**

(a) Given any $x \in \mathbb{C}^n$, find positive constants $c_1$ and $c_2$, independent of $x$ such that

$$c_1 \| x \|_2 \leq \| x \|_{\infty} \leq c_2 \| x \|_2.$$  

(b) If $A \in \mathbb{C}^{m \times n}$, prove that $\| A \|_2 = \sigma_1$, where $\sigma_1$ is the largest singular value of $A$.

(c) Assume that $A \in \mathbb{C}^{m \times n}$ has rank $\tau$. Find a scalar $\sigma$ ($\sigma > 0$), independent of $p$, such that

$$\| Ap \|_{\infty} \geq \sigma \| p \|_2 \quad \text{for all } p \in \text{range}(A^T).$$

**Question 1.2.**

(a) State the **standard rounding-error model** for floating-point arithmetic.

(b) Let $u$ denote the unit round-off. Let $n$ be a positive integer such that $nu < 1$. If $\{ \delta_i \}$ is a set of $n$ numbers such that $|\delta_i| \leq u$, and $\{ s_i \}$ are integers such that $s_i = \pm 1$, prove that

$$\prod_{i=1}^{n} (1 + \delta_i)^{s_i} = 1 + \theta_n,$$

where $|\theta_n| \leq \gamma_n$, with $\gamma_n = nu/(1 - nu)$.

(c) Given two $n$-vectors $x$ and $y$, let $\hat{Z}$ denote the **computed** version of the rank-one matrix $Z = xy^T$. Apply the standard rounding error model to derive a bound for the component-wise forward error in $\hat{Z}$ as an approximation to $Z$. Is the calculation of $\hat{Z}$ backward stable? Justify your answer.

**Question 1.3.** Assume that $A \in \mathbb{R}^{n \times n}$.

(a) Suppose that $\tau$ ($\tau < n$) steps of Householder reduction with column interchanges gives the decomposition

$$AP = Q \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix},$$

where $Q$ is orthogonal, $P$ is a permutation and $R_{11}$ is an $r \times r$ nonsingular upper triangle. Define bases for null($A$) and range($A^T$) in terms of the $QR$ factors above. Verify that the proposed bases satisfy the properties of a basis.

(b) Now assume that $\tau$ steps of Householder reduction give:

$$AP = Q \begin{pmatrix} R_{11} & R_{12} \\ 0 & E \end{pmatrix},$$

where $Q$ is orthogonal, $P$ is a permutation and $R_{11}$ is an $r \times r$ nonsingular upper triangle. Show that $\sigma_n$, the smallest singular value of $A$, satisfies $\sigma_n \leq \| E \|_2$. Give a brief discussion of the implication of this result.
2. Nonlinear Equations, Nonlinear Least-Squares and Optimization

Question 2.1.

(a) Let $F : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ be continuously differentiable on the open convex set $D$. Compute the Fréchet derivative for the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $f(x) = \|x\|_2$.

(b) Given a real $n \times n$ symmetric matrix $A$, find the Fréchet derivative of the function $G : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$ such that
\[
G(x, \lambda) = \begin{pmatrix} Ax - \lambda x \\ \|x\|_2 - 1 \end{pmatrix}.
\]

Hence define an iteration of Newton's method for finding an eigenvalue of $A$ and its associated eigenvector.

(c) An eigenvalue of a matrix is simple if it has algebraic multiplicity 1. If $\lambda^*$ is a simple eigenvalue of $A$ and $x^*$ is its corresponding normalized eigenvector, prove that $G'(x^*, \lambda^*)$ is nonsingular. Give a brief discussion of the implication of this result when finding $x^*$ and $\lambda^*$ using Newton's method.

Question 2.2. Consider the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that
\[
f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.
\]

(a) Compute the spectral decomposition of the Hessian matrix of second derivatives at $\bar{x} = (0, 1, 0)^T$.

(b) Compute the Newton direction $p^N$ and modified Newton direction $p^M$ at $\bar{x}$. Determine if $p^N$ and $p^M$ are descent directions.

(c) Find a direction of negative curvature that is a direction of decrease for $f$ at $\bar{x}$.

Question 2.3.

(a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where $\gamma$ is a scalar and $u$ and $v$ are $n$ vectors.

(b) Given an $n \times n$ symmetric positive-definite matrix $B$, and $n$-vectors $y$ and $s$, consider the symmetric rank-one quasi-Newton update
\[
B_+ = B + \frac{1}{(y - Bs)^T s}(y - Bs)(y - Bs)^T.
\]

(i) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a quadratic function with a symmetric positive-definite Hessian matrix. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$, where $\nabla f(x)$ is the gradient of $f$ evaluated at $x$. If vectors $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{y} = \nabla f(\bar{x}_+) - \nabla f(\bar{x})$ satisfy $B\bar{s} = \bar{y}$, prove that $B_+\bar{s} = \bar{y}$.

(ii) Find a condition on the vectors $y$ and $s$ that will guarantee the positive definiteness of $B_+$. 
3. **Approximation and Numerical ODEs**

In this part, we assume that \( a, b \in \mathbb{R} \) with \( a < b \). We also denote by \( \mathcal{P}_n \) the set of all polynomials of degree \( \leq n \) for any integer \( n \geq 0 \).

**Question 3.1.**

(a) Prove for any \( f \in C[a, b] \) that

\[
\lim_{n \to \infty} \inf_{q_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - q_n(x)| = 0,
\]

\[
\lim_{n \to \infty} \inf_{q_n \in \mathcal{P}_n} \int_a^b [f(x) - q_n(x)]^2 \, dx = 0.
\]

(b) Let \( p_2 \in \mathcal{P}_2 \) be the best uniform approximation in \( \mathcal{P}_2 \) of the function \( g(x) = x^3 - 2x^2 + 1 \) with respect to the \( C[-1,1] \)-norm. What is the value of \( p_2(1) \)? Justify your answer.

(c) Let \( Q_0, \ldots, Q_n, \ldots \) be orthogonal polynomials in \( L^2[a, b] \). Fix \( n \geq 1 \). Prove that \( Q_n \) has \( n \) simple roots in \( [a, b] \).

**Question 3.2.**

(a) Find the degree of precision of the numerical quadrature

\[
\int_a^b f(x) \, dx \approx \frac{1}{2}(b-a)[f(a) + f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)] \quad \forall f \in C^1[a, b].
\]

(b) Consider a sequence of interpolatory numerical integration formulas

\[
\int_a^b f(x) \, dx \approx \sum_{k=1}^n A_k^{(n)} f(x_k^{(n)}), \quad n = 1, \ldots.
\]

Suppose all the coefficients \( A_k^{(n)} \) \((k = 1, \ldots, n; n = 1, \ldots)\) are positive. Prove that

\[
\lim_{n \to \infty} \sum_{k=1}^n A_k^{(n)} f(x_k^{(n)}) = \int_a^b f(x) \, dx \quad \forall f \in C[a, b].
\]