Name ________________________________ SID ________________________________

Qualifying Exam in Numerical Analysis

Fall 2023

Time: 1:00 pm – 4:00 pm, Friday, September 8, 2023

	Full Scores	Your Scores
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Exam Rules and Instructions

- **This is a three-hour, close-book, and close-note exam. No calculators, computers, tablets, and any other electronic devices are allowed. No cheatsheets are allowed.**
- **There are a total of 10 pages (including this cover page) of the exam.**
- **You must show all the computational steps for how you get the answers. No credit will be given for unsupported answers.**
- **All numbers in computational results must be exact, whether written in rational/radical or decimal format. No credit will be given for rounded numbers.**

1. (25 points) Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Consider the linear system $Ax = b$ and the perturbed system $(A + \delta A)\hat{x} = b + \delta b$. Let $\delta x = \hat{x} - x$. Show that

$$\frac{\|\delta x\|_2}{\|\hat{x}\|_2} \le \|A\|_2\|A^{-1}\|_2 \cdot \left(\frac{\|\delta A\|_2}{\|A\|_2} + \frac{\|\delta b\|_2}{\|A\|_2\|\hat{x}\|_2} \right).$$

Also show that there exist δA and $\delta b = 0$ such that the above is an equality.

2. (25 points) Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$. Write an algorithm for computing orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that the product $B := U^T AV$ is an m-by-n upper bidiagonal matrix, i.e., $B_{ij} = 0$ if either $i > j$ or $j > i + 1$.

3. (25 points) Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Show that the Jacobi iterative method for solving the linear system $Ax = b$ converges for all initial points $x_0 \in \mathbb{R}^n$ if A is strictly row diagonally dominant.

4. (25 points) Let $n \geq 1$ be an integer and $f(x) = x^{n+2}$ for all $x \in [-1, 1]$. Find the best uniform approximation of f in $\mathcal{P}_n$ (the class of real polynomials of degree $\leq n$) with respect to the $C([-1, 1])$-norm. (You need to find an explicit formula of the approximation and justify your answer.)

5. (25 points) Let $f \in C^3([0,1])$. Let $n \geq 2$ be an integer and $x_i = i/n$ $(i = 0, 1, \ldots, n)$. Define $I_n f : [0,1] \to \mathbb{R}$ as follows: on the ith interval $[x_{i-1}, x_i]$ $(1 \leq i \leq n)$, $I_n f$ is the Lagrange interpolation polynomial of degree ≤ 2 that interpolates f at $x_{i-1}, (x_{i-1} + x_i)/2, x_i$. Denote $M_3 = \max_{x \in [0,1]} |f^{(3)}(x)|$.

(1) Prove that $I_n f \in C([0,1])$ and that $\max\limits_{x \in [0,1]} |I_n f(x) - f(x)| \leq \dfrac{M_3}{6n^3}$.

(2) Can you improve the above bound by replacing $1/6$ by a smaller number, as smaller as possible?

6. (25 points) Let $n \geq 1$ be an integer, $x_1, \ldots, x_n \in [-1, 1]$ be n distinct points, and $w_1, \ldots, w_n$ be n real numbers. Denote by d_n the degree of precision of the numerical quadrature

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{n} w_k f(x_k).$$

(1) Prove that $d_n \leq 2n - 1$.

(2) Define $l_k(x) = \prod_{1 \leq j \leq n, j \neq k} (x - x_j)/(x_k - x_j)$ $(k = 1, \ldots, n)$. Assume further that the numbers $w_1, \ldots, w_n$ are given by $w_k = \int_{-1}^{1} l_k(x)\,dx$ $(k = 1, \ldots, n)$. Prove that $d_n \geq n - 1$.

(3) Assume further that $\omega_n(x) = (x - x_1) \cdots (x - x_n)$ is an nth orthogonal polynomial on $[-1, 1]$, i.e., $\int_{-1}^{1} \omega_n(x) q(x)\,dx = 0$ for any polynomial of $q(x)$ of degree $\leq n - 1$. Prove that $d_n = 2n - 1$.

7. (25 points) Consider the initial value problem with ODE

$$y' = f(t, y)$$

and initial value $y(t_0) = y_0$.

(a) Briefly describe the steps involved to generate the difference formula for the k-step Adams-Bashforth method, where $k \geq 1$.

(b) Use this to derive the actual difference formula for the 2-step Adams-Bashforth method.

8. (25 points) Consider the Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

for solving the initial value problem with ODE

$$y' = f(t, y)$$

and initial value $y(t_0) = y_0$. Determine whether this method is A-stable. Be sure to justify your conclusion.