MATH 240 Qualifying Exam

Sept. 12, 2024

Instructions: 3 hours, open book/notes (only Folland or personal lecture notes; no HW or other solutions). You may use without proofs results proved in Folland Chapters 1-8. Present your solutions clearly, with appropriate detail.

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1. (25 pts) Let $E \subseteq \mathbb{R}$ be (Lebesgue) measurable and satisfy $E + r = E$ for every rational number r. Show that either E or E^c has measure 0.

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2. (40 pts) For any $n \in \mathbb{N}$, find all Lebesgue measurable sets $E \subseteq \mathbb{R}^n$ such that there exists some $f_E \in L^1(\mathbb{R}^n)$ satisfying

$$\lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f_E(y)dy = \infty \qquad \text{for each } x \in E.$$

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3. (30 pts) Let X be a topological vector space. A net (or, less generally, a sequence) $\langle x_\alpha \rangle_{\alpha \in A}$ in X is *Cauchy* if the net of pairwise differences $\langle x_\alpha - x_\beta \rangle_{(\alpha,\beta) \in A \times A}$, with $A \times A$ directed by the rule $(\alpha, \beta) \precsim (\alpha', \beta') \Leftrightarrow (\alpha \precsim \alpha'$ and $\beta \precsim \beta')$, converges to $0 \in X$. Prove that if X is first countable and every Cauchy sequence in X converges, then every Cauchy net in X converges.

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4. (25 pts) Recall that the Riemann-Lebesgue lemma implies that for all $f \in L^1([0,1], m)$,

$$\lim_{n \to \infty} \int_0^1 f(x) e^{inx} \, dx = 0.$$

Show that there is no rate of convergence for the above limit that is independent of f. Specifically, show that there cannot exist any sequence of positive numbers (a_n) such that $a_n \to \infty$ and for every $f \in L^1([0,1], m)$ there is a constant $C_f < \infty$ with

$$\left| \int_0^1 f(x) e^{inx} \, dx \right| \le \frac{C_f}{a_n} \qquad \text{for each } n \in \mathbb{N}.$$

Hint: If such (a_n) existed, find a sequence of linear functionals that yields a contradiction.

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5. (40 pts) Let X be an LCH space.

(a) If μ is a Radon measure on X satisfying $\mu(X) = \infty$, prove that there is a non-negative function $f \in C_0(X)$ satisfying $\int f \, d\mu = \infty$.

(b) Prove that every positive linear functional on $C_0(X)$ is bounded.

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6. (25 pts) Show that for any $p > 1$, there is a constant $C_p < \infty$ such that

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)| \le \|f\|_{L^1} + C_p \|f'\|_{L^p} \qquad \text{for each } f \in C^1(\mathbb{T}).$$