Name:	____________________
Student #:	____________________

Some notations and identities you may use

1. $\tau_y f(x) = f(x-y)$, $f * g(x) = \int f(x-y)g(y)\,dy$
2. For $f \in L^1(\mathbb{R}^n, m)$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi\sqrt{-1}\langle \xi, y\rangle} f(y)\,dy$
3. m and dy denote the Lebesgue measure, $\check{f}(x) = \hat{f}(-x)$
4. For $\lambda > 0$, $g^\lambda(x) = e^{-\pi\lambda|x|^2}$, $\widehat{g^\lambda}(\xi) = \lambda^{-n/2} e^{-\pi|\xi|^2/\lambda}$
5. You may quote a result from the book or lecture by stating the result clearly or by the name (such as *the monotone convergence theorem*).

Problem	Points
Page 2 (30 points)	
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Total (130 points)	

(1) (10+10+10 pts) TRUE or FALSE: If true, prove it. If false, disprove it.

(a) If f is a linear functional of a normed vector space $\mathcal{X}$, $f^{-1}(0)$ is closed.

(b) In a Hilbert space, if $\{x_n\}$ converges to x weakly and $\|x_n\| \to \|x\|$, then $\{x_n\}$ converges to x strongly, namely $\|x_n - x\| \to 0$.

(c) Let $E \subset \mathbb{R}$ be Lebesgue measurable set and assume that there exists $0 < \alpha < 1$ such that $m(E \cap I) \leq \alpha m(I)$ for all open intervals I. Then, $m(E) = 0$.

(2) (3+7) Assume that $\mu(X) < \infty$. Let $\{f_n\}$ be a bounded sequence of complex functions. Assume that $f_n \to f$ uniformly as $n \to \infty$. Prove that $\int_X f_n \, d\mu \to \int_X f d\mu$. Show by an example that the assumption $\mu(X) < \infty$ can not be dropped.

(3) (10 pts) Let $\mathbb{R}_+ = [0, \infty)$, $f, g \in L^1(\mathbb{R}_+, m)$, and consider

$$h(x) = \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{dy}{y}.$$

Show that h is well-defined (i.e., $y \to f(y)g(x/y)/y$ is in $L^1(\mathbb{R}_+, m)$) for a.e. $x \in \mathbb{R}_+$, $h \in L^1(\mathbb{R}_+)$, and

$$\|h\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$

Comment: You may use without proof that $g(x/y)$ is Lebesgue measurable on $\mathbb{R}_+^2$.

(4) (7+3+5+15 pts) Define the distance function between (x_1, y_1) and (x_2, y_2) for two points (where x_i, y_i are real numbers) in the plane to be

$$|y_1 - y_2| \text{ if } x_1 = x_2; \quad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2.$$

(i) Prove that this is indeed a metric.
(ii) The corresponding metric space (X, d) so defined is locally compact.
(iii) For any $f \in C_c(X)$, Let F be the set of x such that there exists a y with $f(x, y) \neq 0$. Prove that F is a finite set $\{x_1, x_2, \cdots, x_n\}$.
(iv) For f in (iii) define $I(f) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x_i, y)\, dy$. Then $I(f)$ induces a Radon measure μ on X. Is μ inner regular for all Borel set? If answer is a 'Yes' prove it. if the answer is a 'No' find a Borel set which is not inner regular.

(5) (8+7 pts) Let ℓ^∞ denote the vector space of sequence of complex numbers $x = (x_1, x_2, \cdots, x_n, \cdots)$ with $\|x\|_\infty := \sup_n |x_n| < \infty$. Define $\phi_n(x) = \frac{1}{n}\sum_{k=1}^{n} x_k$. Prove that (i) $\phi_n \in (\ell^\infty)^*$ and $\{\phi_n\}$ has a weak* cluster point ϕ; (ii) ϕ is an element of $(\ell^\infty)^*$ which does not arise from an element of ℓ^1. Here ℓ^1 denotes the normed vector space of sequence $x = (x_1, x_2, \cdots, x_n, \cdots)$ with $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$.

(6) (15 pts) Let $1 \leq p < \infty$. Recall that $\lambda_g(\alpha) = \mu(\{x \,|\, |g(x)| > \alpha\})$. Assume that T is a linear operator from L^p into L^{q_1} and L^{q_2} with $1 \leq q_1 < q_2$ such that $\lambda_{Tf}(\alpha) \leq (C_1\|f\|_p/\alpha)^{q_1}$ and $\lambda_{Tf}(\alpha) \leq (C_2\|f\|_p/\alpha)^{q_2}$. Prove that for any $q_1 < q < q_2$, $\|Tf\|_q \leq C_q\|f\|_p$. Here C_q depends on q, q_1, q_2 and C_1, C_2.

(7) (20 pts) Let $\Gamma(z)$ be the gamma function which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

for z with $\Re(z) > 0$. For a compact support function ϕ prove that for any $0 < \alpha < n$

$$\frac{\Gamma((n-\alpha)/2)}{\pi^{(n-\alpha)/2}} \int_{\mathbb{R}^n} |x|^{\alpha-n} \hat{\phi}(x)\, dx = \frac{\Gamma(\alpha/2)}{\pi^{\alpha/2}} \int_{\mathbb{R}^n} |\xi|^{-\alpha} \phi(\xi)\, d\xi.$$

The $dx, d\xi$ are all with respect to the Lebesgue measure of the corresponding Euclidean spaces.

Hint: Use the Fourier transform of the Gaussian (on the covering page), the identify $\int \hat{f}\, g = \int f \hat{g}$ for L^1 functions and the change of variables for integral in the definition of Γ function.

END OF EXAM