Some notations and identities you may use

1. $\tau_y f(x) = f(x - y)$, $f * g(x) = \int f(x - y)g(y)\,dy$
2. For $f \in L^1(\mathbb{R}^n, m)$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi \sqrt{-1} \langle \xi, y \rangle} f(y)\,dy$
3. $m$ and $dy$ denote the Lebesgue measure, $\hat{f}(x) = \hat{f}(-x)$
4. For $\lambda > 0$, $g^\lambda(x) = e^{-\pi \lambda |x|^2}$, $\hat{g}(\xi) = \lambda^{-n/2} e^{-\pi |\xi|^2 / \lambda}$
5. You may quote a result from the book or lecture by stating the result clearly or by the name (such as the monotone convergence theorem).
(1) (10+10+10 pts) TRUE or FALSE: If true, prove it. If false, disprove it.
(a) If $f$ is a linear functional of a normed vector space $X$, $f^{-1}(0)$ is closed.

(b) In a Hilbert space, if $\{x_n\}$ converges to $x$ weakly and $\|x_n\| \to \|x\|$, then $\{x_n\}$ converges to $x$ strongly, namely $\|x_n - x\| \to 0$.

(c) Let $E \subset \mathbb{R}$ be Lebesgue measurable set and assume that there exists $0 < \alpha < 1$ such that $m(E \cap I) \leq \alpha m(I)$ for all open intervals $I$. Then, $m(E) = 0$. 
(2) (3+7) Assume that \( \mu(X) < \infty \). Let \( \{f_n\} \) be a bounded sequence of complex functions. Assume that \( f_n \to f \) uniformly as \( n \to \infty \). Prove that \( \int_X f_n \, d\mu \to \int_X f \, d\mu \). Show by an example that the assumption \( \mu(X) < \infty \) can not be dropped.
(3) (10 pts) Let \( \mathbb{R}_+ = [0, \infty) \), \( f, g \in L^1(\mathbb{R}_+, m) \), and consider

\[ h(x) = \int_0^\infty f(y) g \left( \frac{x}{y} \right) \frac{dy}{y}. \]

Show that \( h \) is well-defined (i.e., \( y \to f(y)g(x/y)/y \) is in \( L^1(\mathbb{R}_+, m) \)) for a.e. \( x \in \mathbb{R}_+ \), \( h \in L^1(\mathbb{R}_+) \), and

\[ \| h \|_{L^1} \leq \| f \|_{L^1} \| g \|_{L^1}. \]

Comment: You may use without proof that \( g(x/y) \) is Lebesgue measurable on \( \mathbb{R}_+^2 \).
(4) (7+3+5+15 pts) Define the distance function between \((x_1, y_1)\) and \((x_2, y_2)\) for two points (where \(x_i, y_i\) are real numbers) in the plane to be
\[
|y_1 - y_2| \text{ if } x_1 = x_2; \quad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2.
\]
(i) Prove that this is indeed a metric.
(ii) The corresponding metric space \((X, d)\) so defined is locally compact.
(iii) For any \(f \in C_c(X)\), Let \(F\) be the set of \(x\) such that there exists a \(y\) with \(f(x, y) \neq 0\). Prove that \(F\) is a finite set \(\{x_1, x_2, \ldots, x_n\}\).
(iv) For \(f\) in (iii) define \(I(f) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x_i, y) \, dy\). Then \(I(f)\) induces a Radon measure \(\mu\) on \(X\). Is \(\mu\) inner regular for all Borel set? If answer is a ‘Yes’ prove it. if the answer is a ‘No’ find a Borel set which is not inner regular.
(5) (8+7 pts) Let $\ell^\infty$ denote the vector space of sequence of complex numbers $x = (x_1, x_2, \cdots, x_n, \cdots)$ with $\|x\|_\infty := \sup_n |x_n| < \infty$. Define $\phi_n(x) = \frac{1}{n} \sum_{k=1}^n x_k$. Prove that (i) $\phi_n \in (\ell^\infty)^*$ and $\{\phi_n\}$ has a weak* cluster point $\phi$; (ii) $\phi$ is an element of $(\ell^\infty)^*$ which does not arise from an element of $\ell^1$. Here $\ell^1$ denotes the normed vector space of sequence $x = (x_1, x_2, \cdots, x_n, \cdots)$ with $\|x\|_1 = \sum_{k=1}^\infty |x_k|$. 
(6) (15 pts) Let $1 \leq p < \infty$. Recall that $\lambda_g(\alpha) = \mu(\{x \mid |g(x)| > \alpha\})$. Assume that $T$ is a linear operator from $L^p$ into $L^{q_1}$ and $L^{q_2}$ with $1 \leq q_1 < q_2$ such that $\lambda_{Tf}(\alpha) \leq (C_1\|f\|_p/\alpha)^{q_1}$ and $\lambda_{Tf}(\alpha) \leq (C_2\|f\|_p/\alpha)^{q_2}$. Prove that for any $q_1 < q < q_2$, $\|Tf\|_q \leq C_q\|f\|_p$. Here $C_q$ depends on $q, q_1, q_2$ and $C_1, C_2$. 
(7) (20 pts) Let \( \Gamma(z) \) be the gamma function which is defined by
\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt
\]
for \( z \) with \( \Re(z) > 0 \). For a compact support function \( \phi \) prove that for any \( 0 < \alpha < n \)
\[
\frac{\Gamma((n - \alpha)/2)}{\pi^{(n-\alpha)/2}} \int_{\mathbb{R}^n} |x|^{\alpha-n} \hat{\phi}(x) \, dx = \frac{\Gamma(\alpha/2)}{\pi^{\alpha/2}} \int_{\mathbb{R}^n} |\xi|^{-\alpha} \phi(\xi) \, d\xi.
\]
The \( dx, d\xi \) are all with respect to the Lebesgue measure of the corresponding Euclidean spaces.

Hint: Use the Fourier transform of the Gaussian (on the covering page), the identify \( \int \hat{f} \hat{g} = \int f \hat{g} \) for \( L^1 \) functions and the change of variables for integral in the definition of \( \Gamma \) function.

END OF EXAM