Name:	____________________
Student #:	____________________

Some notations and identities you may use

1. $\tau_y f(x) = f(x-y)$, $f * g(x) = \int f(x-y)g(y)\,dy$
2. For $f \in L^1$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi\sqrt{-1}\langle \xi, y\rangle} f(y)\,dy$
3. $\check{f}(x) = \hat{f}(-x)$
4. For $\lambda > 0$, $g^\lambda(x) = e^{-\pi\lambda|x|^2}$, $\hat{g^\lambda}(\xi) = \lambda^{-n/2}e^{-\pi|\xi|^2/\lambda}$
5. Lesbegue measure on Euclidean space is denoted as dy (above in 1. 2.), m or dm.
6. You may quote a result from the textbook (Folland) or lecture by stating the assumption and conclusion correctly and clearly or by the name (such as *the monotone convergence theorem*).

Problem	Points
Page 2 (30 points)	
Page 3 (15 points)	
Page 4 (10 points)	
Page 5 (15 points)	
Page 6 (20 points)	
Page 7 (20 points)	
Page 8 (20 points)	
Total (130 points)	

(1) (30=15+15) TRUE or FALSE: Prove it if true and disprove it if false

(i) Let $f(t)$ be a monotone non-increasing function on $\mathbb{R}$. Then its distributional derivative is always a Radon measure.

(ii) Let $E \subset [0,1] \subset \mathbb{R}$ be a countable subset. Then, for any $\epsilon > 0$, there is a finite *cover of E by open intervals $\{I_k\}_{k=1}^n$ such that*

$$\sum_{k=1}^{n} m(I_k) < \epsilon.$$

(2) (15) For $a > 0$, let $(S_a f)(x) = f(x/a)$ for Lebesgue measurable functions f on $\mathbb{R}$. Then for any $f \in L^1(\mathbb{R}, m)$, $S_a f \to f$ in L^1 as $a \to 1$.

(3) (10) Let X be a σ-compact LCH. Let μ be a Radon measure on X. Let $f \geq 0$ be a measurable function. Prove that if for any open subset U such that $\mu(U) = \int_U f\, d\mu$. Then $f = 1$ μ- a.e.

(4) (5+5+5) Let $\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$ (with $\operatorname{sinc} 0 = 1$). Prove

(i) If $a > 0, \hat{\chi}_{[-a,a]} = \check{\chi}_{[-a,a]} = 2a \operatorname{sinc} 2ax$.

(ii) Let $\mathcal{H}_a = \{f \in L^2, \hat{f}(\xi) = 0, \text{ if } |\xi| > a\}$. Then $\mathcal{H}_a$ is a Hilbert space and $\{\sqrt{2a} \operatorname{sinc}(2ax - k), k \in \mathbb{Z}\}$ is an orthonormal basis.

(iii) If $f \in \mathcal{H}_a$, then $f \in C_0$ (namely continuous function which vanishes at infinity) and $f = \sum_{-\infty}^{\infty} f(\frac{k}{2a}) \operatorname{sinc}(2ax - k)$ in L^2.

(5) (20) Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space satisfying: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. Prove that the space is a Hilbert space and the norm is induced by the inner product. Namely you need to construct an inner product $(\cdot, \cdot)$ on $\mathcal{X}$ such that its induced norm is the same as $\|\cdot\|$.

(6) (20=10+10) (i) Prove that for $p \geq 1$ $f \in L^p$ if and only if $\sum_{-\infty}^{+\infty} \beta^{kp} \lambda_f(\beta^k) < \infty$ for all $\beta > 1$. Here $\lambda_f(\alpha) = \mu(\{x | |f|(x) > \alpha\})$.

(ii) Assume that T is a linear operator from L^p into L^{q_1} and L^{q_2} with $1 < q_1 < q_2$ such that $\lambda_{Tf}(2^k) \leq \left(C_1 \|f\|_p / 2^k\right)^{q_1}$ for integers $k \leq 0$; and $\lambda_{Tf}(2^\ell) \leq \left(C_2 \|f\|_p / 2^\ell\right)^{q_2}$, for integers $\ell \geq 0$. Prove that for any $q_1 < q < q_2$, $\|Tf\|_q \leq C_q \|f\|_p$. Here C_q depends on q, q_1, q_2 and C_1, C_2.

(7) (20) Let (X, μ) be a nonempty measurable space with $\mu(X) < \infty$, $f \in L^\infty(\mu)$ and $\|f\|_\infty > 0$. Define $\alpha_n := \int_X |f|^n$ for $n = 1, 2, 3 \cdots$. Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

END OF EXAM