Math281 AB Qualifying Exam (60 points)
Spring 2023 Name: _____

Please provide detailed proofs. Failure to include necessary conditions, neglecting to check whether the conditions hold, and applying theorems without explanation will result in penalties. Any texts copied from online resources will receive a grade of zero. Small variations of blog posts or Q&A webpages will also be graded as zero, and the student will be reported to the Academic Integrity Office for further action. Additionally, all UCSD integrity rules apply in conjunction with the aforementioned policy.

Consider a panel dataset consisting of N individuals observed over T time periods. The dataset can be represented as X_{it}, where i denotes the individual index and t denotes the time index. We aim to estimate an unknown parameter θ using the Huber estimator, denoted as $\hat{\theta}_n$. The model can be represented as:

$$X_{it} = \theta + \alpha_i + \epsilon_{it},$$

where X_{it} is the observed data for individual i at time t, θ is the parameter of interest, α_i represents the fixed effect for individual i, and ϵ_{it} is the error term. The α_i's and ϵ_{it}'s respectively represent the unobservable individual specific effects and identically distributed (i.i.d.) error terms with $E[\epsilon_{it}|\alpha_i] = 0$ and $E[\epsilon_{it}^2|\alpha_i] = \mathbb{I}_T$, i.e. the identity matrix and $E[\epsilon_{it}\epsilon_{is}|\alpha_i] = 0$ for $t \neq s$.

1. (15 points) Set up a least squares M-estimator for θ that would be independent of the unknown parameters α_i. Hint: consider an appropriate data-transformation.

2. (10 points) Set up a Huber estimator for $\hat{\theta}_n$ for estimating θ. Huber's estimator is an M-estimator with the following loss function

$$\rho(x) = \begin{cases} \frac{1}{2}x^2 & \text{for } |x| \leq k \\ k|x| - \frac{1}{2}k^2 & \text{for } |x| > k \end{cases} \tag{1}$$

   In the above k is a fixed number. You can consider $k = 1$.

3. (15 points) Then prove the consistency of the Huber's estimator.

4. (20 points) Then prove the asymptotic normality of the Huber's estimator as both N and T go to infinity.