1. Let $X_1, X_2, \ldots, X_n$ be an iid sample with pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma x}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$, with $x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. Consider the following estimator of $\mu$:

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{n} T_i}{n - 1} + 1,$$

where we define $T_i = \ln X_i$. It is well known that $T_i \sim N(\mu, \sigma^2)$ (do not show this).

(a) Show that the bias of $\hat{\mu}_1$ is

$$\frac{\mu}{n - 1} + 1.$$

(b) Show that the variance of $\hat{\mu}_1$ is

$$\frac{n\sigma^2}{(n - 1)^2}.$$

(c) Deduce the mean squared error of the estimator. Is $\hat{\mu}_1$ a consistent estimator of $\mu$? Why?

(d) Transform $\hat{\mu}_1$ to obtain an unbiased estimator $\hat{\mu}_2$ of $\mu$.

2. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from the pdf

$$f(x, \theta) = \theta (1 + x)^{-\theta - 1},$$

with $0 < x < \infty$ and $\theta > 0$.

(a) Show that the maximum likelihood estimator of $\theta$ is

$$\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} \ln(1 + X_i)}.$$

(b) Show that the Fisher information $I_n(\theta)$ is equal to $n/\theta^2$.

(remember that $I_n(\theta)$ is minus the expectation of the second derivative of the log likelihood).
3. Suppose \( X_1, X_2, \ldots, X_n \) are iid from a distribution with pdf \( f(x|\theta) \) where \( T \) is a sufficient statistic. Also, assume the Bayesian framework with prior pdf's \( \pi(\theta|\gamma) \) and \( \psi(\gamma) \). Show that the posterior distribution depends on the data only through \( T \).

4. Suppose that

\[
X \sim f(x|\lambda) = \frac{\lambda^3}{2} x^2 e^{-\lambda x}; \quad x \geq 0 \text{ and } \lambda > 0 (\lambda \text{ a constant}).
\]

Give conditions on \( g(x) \) under which

\[
E_\lambda g'(X) = \lambda E_\lambda g(X) - E_\lambda \left( \frac{2g(X)}{X} \right).
\]

5. Let \( X_1, X_2, \ldots, X_n \) be an iid sample from a distribution \( F_X \) and let \( Y_1, Y_2, \ldots, Y_n \) be an iid sample from a distribution \( F_Y \), and the two samples are independent. The Wilcoxon-Mann-Whitney test statistic is defined by

\[
T_X = \sum_{i=1}^m r(X_i),
\]

where \( r(X_i) \) denotes the rank of \( X_i \) in the global sample, i.e. in the sample of \( X \)'s and \( Y \)'s together.

Suppose we want to test

\[
H_0 : F_X = F_Y
\]

against

\[
H_1^- : F_X < F_Y.
\]

Let \( t \) be the observed value of the test statistic \( T_X \). Are we going to reject \( H_0 \) if \( t \) is too large or if \( t \) is too small? Justify.

6. Let \( X_1, X_2, \ldots, X_n \) be an iid sample from an unknown continuous density \( f \). Consider the kernel density estimator of \( f \), i.e.

\[
\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n K\left( \frac{x - X_i}{h} \right),
\]

where the kernel \( K \) is bounded and satisfies \( \int K(x) \, dx = 1 \), \( \int xK(x) \, dx = 0 \) and \( 0 < \int x^2 K(x) \, dx < \infty \).
(a) Show that the bias of the estimator is given by

\[ \text{Bias}[\hat{f}(x)] = K_h * f(x) - f(x), \]

where \( K_h(x) = h^{-1} K(x/h) \).

(b) Suppose now that the density \( f \) has two bounded and continuous derivatives. Show that the bias satisfies

\[ \text{Bias}[\hat{f}(x)] = \frac{h^2}{2} f''(x) \int x^2 K(x) \, dx + o(h^2). \]

Hint: use Taylor expansion of \( f \).