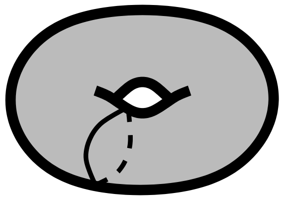
41. Summer 2023

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

1. Let $p : S^3 \to S^2$ be the Hopf mapping, defined by sending $(Z, W) \in S^3 \subseteq \mathbb{C}^2$ to $Z/W \in \mathbb{C} \cup \{\infty\}$. Show that there does not exist a *section*, that is a map $s : S^2 \to S^3$ satisfying $p \circ s = \mathrm{id}_{S^2}$.

2. Let X be the space obtained by gluing the boundary of a disc to the curve in the torus shown below. Compute the second homotopy group $\pi_2(X)$.



3. Let $F_2 = \langle a, b \rangle$ be the free group on 2 letters and let S_3 be the symmetric group. Let K be the kernel of the homomorphism $F_2 \to S_3$ given by sending $a \mapsto (12), b \mapsto (23)$. To which well-known group is K isomorphic?

4. Let M^3 be a closed path-connected non-orientable 3-manifold. Show that its Euler characteristic is 0 and that its fundamental group is infinite.

5. Let X be the space obtained from a solid ball B^3 by identifying pairs of antipodal points on its boundary sphere S^2. Decompose X as a CW-complex and compute its homology $H_*(X; \mathbb{Z})$.

6. Let M^n be a path-connected closed orientable manifold such that there exists a map $f : S^n \to M$ of degree ± 1. Show that $H^*(M; \mathbb{F}) = H^*(S^n; \mathbb{F})$ whenever $\mathbb{F}$ is a field, and hence that $H^*(M; \mathbb{Z}) \cong H^*(S^n; \mathbb{Z})$.

7. Let $f : S^{2n+1} \to S^{2n+1}$ be a map satisfying $f(-x) = -f(x)$. Show that the degree of f must be odd.

8. Let $\mathbb{C}P^2$ be the complex projective plane with its usual orientation, and $\overline{\mathbb{C}P}^2$ be the same manifold with the opposite orientation. Let $M^4 = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ be the connect-sum of the two manifolds, obtained by removing an open 4-ball from each one and identifying the resulting 3-spheres so that the result is oriented. By working out the cohomology ring $H^*(M; \mathbb{Z})$, show that M is not homotopy-equivalent to $S^2 \times S^2$.