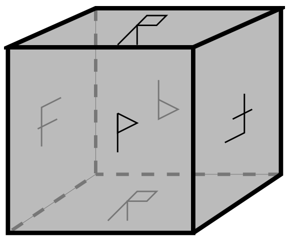
42. Fall 2023

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

1. Suppose that M is a closed connected m-manifold, that N is a closed connected n-manifold, and that M and N are homotopy-equivalent. Show that $m = n$ and that M is orientable if and only if N is orientable. What happens if M and N are merely compact manifolds-with-boundary - that is, they are not necessarily closed?

2. Let X be the space obtained by identifying the faces of a standard cube I^3 in pairs, as shown below. Compute the integral homology groups $H_*(X;\mathbb{Z})$.



3. The torus T, the Klein bottle K and the connect-sum $C = \mathbb{R}P^2 \# \mathbb{R}P^2$ all have isomorphic mod-2 homology groups. Compute the intersection forms (on the first homology with mod-2 coefficients) of these three spaces; to what extent are they distinguishable using these intersection forms?

4. Show that a map $f : S^n \to S^n$ which has no fixed points must be homotopic to the antipodal map. Use this to deduce that a non-trivial finite group acting freely on S^{2n} must be isomorphic to $\mathbb{Z}_2$.

5. Let K be the Klein bottle. Compute the homology groups $H_*(K \times K;\mathbb{Z})$ and cohomology groups $H^*(K \times K;\mathbb{Z})$ (you don't need to work out the ring structure on the cohomology).

6. Show that for any $n \geq 1$ the natural quotient map $S^n \to \mathbb{R}P^n$ is not null-homotopic.

7. Suppose that $S^3 = M \cup_\Sigma N$ is a decomposition of the 3-sphere into two compact 3-manifolds-with-boundary, glued along their common boundary surface Σ. Prove that $H^1(N;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$. Let X be the compact 3-manifold-with-boundary obtained by removing the interior of a small ball from $\mathbb{R}P^3$. Conclude that X cannot be embedded in S^3.

8. Let G be a finite group of order d acting freely on a space X, so that we have a covering map $\pi : X \to Y = X/G$. By lifting singular simplexes, construct a chain map $\tau_* : C_*(Y;\mathbb{Z}) \to C_*(X;\mathbb{Z})$ such that the composite

$$C_*(Y;\mathbb{Z}) \xrightarrow{\tau_*} \mathbb{C}_*(X;\mathbb{Z}) \xrightarrow{\pi_*} C_*(Y;\mathbb{Z})$$

is multiplication by d. Use this to show that we can identify the rational homology of Y with the G-invariant subspace of the rational homology of X:

$$H_*(Y;\mathbb{Q}) \cong H_*(X;\mathbb{Q})^G.$$