 Qualifying Exams

42. Fall 2023

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

1. Suppose that $M$ is a closed connected $m$-manifold, that $N$ is a closed connected $n$-manifold, and that $M$ and $N$ are homotopy-equivalent. Show that $m = n$ and that $M$ is orientable if and only if $N$ is orientable. What happens if $M$ and $N$ are merely compact manifolds-with-boundary - that is, they are not necessarily closed?

2. Let $X$ be the space obtained by identifying the faces of a standard cube $I^3$ in pairs, as shown below. Compute the integral homology groups $H_*(X; \mathbb{Z})$.

![Diagram of a space obtained by identifying the faces of a standard cube](image)

3. The torus $T$, the Klein bottle $K$ and the connect-sum $C = \mathbb{R}P^2 \# \mathbb{R}P^2$ all have isomorphic mod-2 homology groups. Compute the intersection forms (on the first homology with mod-2 coefficients) of these three spaces; to what extent are they distinguishable using these intersection forms?

4. Show that a map $f : S^n \to S^n$ which has no fixed points must be homotopic to the antipodal map. Use this to deduce that a non-trivial finite group acting freely on $S^{2n}$ must be isomorphic to $\mathbb{Z}_2$.

5. Let $K$ be the Klein bottle. Compute the homology groups $H_*(K \times K; \mathbb{Z})$ and cohomology groups $H^*(K \times K; \mathbb{Z})$ (you don’t need to work out the ring structure on the cohomology).

6. Show that for any $n \geq 1$ the natural quotient map $S^n \to \mathbb{R}P^n$ is not null-homotopic.

7. Suppose that $S^3 = M \cup_{\Sigma} N$ is a decomposition of the 3-sphere into two compact 3-manifolds-with-boundary, glued along their common boundary surface $\Sigma$. Prove that $H^1(N; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$. Let $X$ be the compact 3-manifold-with-boundary obtained by removing the interior of a small ball from $\mathbb{R}P^3$. Conclude that $X$ cannot be embedded in $S^3$.

8. Let $G$ be a finite group of order $d$ acting freely on a space $X$, so that we have a covering map $\pi : X \to Y = X/G$. By lifting singular simplexes, construct a chain map $\tau_* : C_*(Y; \mathbb{Z}) \to C_*(X; \mathbb{Z})$ such that the composite

$$C_*(Y; \mathbb{Z}) \xrightarrow{\pi_*} C_*(X; \mathbb{Z}) \xrightarrow{\tau_*} C_*(Y; \mathbb{Z})$$

is multiplication by $d$. Use this to show that we can identify the rational homology of $Y$ with the $G$-invariant subspace of the rational homology of $X$:

$$H_*(Y; \mathbb{Q}) \cong H_*(X; \mathbb{Q})^G.$$