Summer 2018

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

1. Let S_3 be the symmetric group on 3 letters. If we pick elements $\tau, \sigma \in S_3$ of orders 2 and 3 respectively, we get a surjective homomorphism $\theta : \mathbb{Z}_2 * \mathbb{Z}_3 \to S_3$. By constructing a suitable covering space of a 2-complex, show that the kernel of θ is a free group of rank 2.

2. Let S^2 be the standard unit sphere, and let $R_\theta : S^2 \to S^2$ be the operation of rotation through angle θ anticlockwise about the z-axis. Let M be the closed 4-manifold obtained by gluing together two copies A_1, A_2 of $B^2 \times S^2$ along their common boundary $S^1 \times S^2$; specifically, identify

$$(e^{i\theta}, v) \in \partial A_1 \quad \sim \quad (e^{i\theta}, R_\theta(v)) \in \partial A_2 \qquad \text{for all } e^{i\theta} \in S^1, v \in S^2.$$

Use Mayer-Vietoris to compute $H_*(M; \mathbb{Z})$. Give an example of another closed 4-manifold N with the same homology, and use intersection theory to show that M and N are not homotopy-equivalent.

3. Let X_n be the space formed from the disjoint union of n copies $C_1, \ldots, C_n$ of the cylinder $S^1 \times I$ by gluing, for each k, the $S^1 \times \{1\}$ of C_k to the $S^1 \times \{0\}$ of C_{k+1} using a map of degree k. There is a natural sequence of inclusions $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ and so we may define X to be the direct limit X of this family. (X is called a *mapping telescope*.) What is $H_1(X; \mathbb{Z})$? (You might find it helpful to view X as a CW-complex, but you don't have to.)

4. Let Y be a space obtained by attaching a 4-ball, via a degree 6 map of its boundary, to a 3-sphere. Calculate the integral homology $H_*(Y \times \mathbb{R}P^2; \mathbb{Z})$.

5. Given any map $f : S^n \to S^n$, let $\Gamma_f = \{(x, f(x)) : x \in S^n\} \subseteq S^n \times S^n$ be the graph of f. By using the intersection theory of $S^n \times S^n$, calculate the intersection number $[\Gamma_{\mathrm{id}}].[\Gamma_f]$ and deduce that f must have at least one fixed point provided $\deg f \neq (-1)^{n+1}$,.

6. Let M^3 be a *homology sphere* – a closed 3-manifold having the same homology groups as S^3 – and let $X = \Sigma M$ be its suspension. What are the fundamental group and homology groups of X? Show that X is homotopy-equivalent to S^4.

7. Suppose that $S^3 = M \cup_\Sigma N$ is a decomposition of the 3-sphere into two compact 3-manifolds, glued along their common boundary surface Σ. Prove that $H^1(N) \cong H_1(M)$, and conclude that $\mathbb{R}P^2$ cannot be embedded in S^3.