QUALIFYING EXAMS

September 9 2020
5PM-8PM (Pacific Time)

Three-hour exam. Do as many questions as you can. **No book or notes allowed.** Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly. **Even if you can not solve the whole problem, you still need to write your partial answer to receive partial credit.**

1. Show that there does not exist a continuous map \( f : S^1 \times S^1 \to S^1 \) that satisfies both of the following conditions:
   - \( f(x, x) = x \) for any \( x \in S^1 \);
   - \( f(x, y) = f(y, x) \) for any \( x, y \in S^1 \).

2. Let \( X \) be a path connected CW complex whose fundamental group is finite. Show that any continuous map \( f : X \to T^n \) is null homotopic. (Here \( T^n = S^1 \times \cdots \times S^1 \) is the n-dimensional torus.)

3. Compute the homology group \( H_k(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \) for all \( k \geq 0 \). (You need to show your computation of \( H_k(\mathbb{R}P^2; \mathbb{Z}) \).)

4. For \( n \geq 1 \), show that one can not cover the complex projective space \( \mathbb{C}P^n \) by \( n \) open subsets \( U_1, U_2, \cdots, U_n \) such that each \( U_i \) is contractible. (You may assume the ring structure of \( H^*(\mathbb{C}P^n; \mathbb{Z}) \).)

5. For \( n \geq 1 \), take a point \( p \in S^n \) and consider the following subspace of \( S^n \times S^n \)
   \[ A = \{(x, y) \in S^n \times S^n \mid x = p \text{ or } y = p \}. \]
   Show that there does not exist a retraction of \( S^n \times S^n \) to \( A \). (Namely, show that there does not exist a continuous map \( r : S^n \times S^n \to A \) that fixes \( A \) pointwisely.)

6. Let \( M, N \) be two connected, closed, oriented \( n \)-dimensional manifolds \((n \geq 1)\). Consider a continuous map \( f : M \to N \) that has nonzero mapping degree. Show that the induced map \( f^* : H^k(N; \mathbb{Q}) \to H^k(M; \mathbb{Q}) \) is injective for any \( k \). (Recall: the mapping degree of \( f \) equals \( d \) if \( f_*[M] = d[N] \), where \([M], [N] \) denote the fundamental classes.)

7. Let \( M \) be a closed, orientable \( n \)-dimensional manifold with **nonzero Euler characteristic**. Consider the map \( f : M \times M \to M \times M \) defined by \( f(x, y) = (y, x) \) for any \( x, y \in M \). Show that any map \( g : M \times M \to M \times M \) that is **homotopic** to \( f \) has a fixed point.

8. Let \( X \) be a connected CW complex such that \( \pi_1(X) \) is a nontrivial finite group and \( \pi_k(X) = 0 \) for any \( k \geq 2 \). Show that \( X \) can not be a finite CW complex. (Namely, \( X \) must have infinitely many cells.) **Hint:** Compute the Euler characteristic of the universal covering space.