

# Necessary Conditions For the Non-existence of Odd Perfect Numbers

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## *Introduction*

- For hundreds of years, the many facets of number theory have fascinated mathematicians. One particularly old topic of interest is that of perfect numbers. Since the name may be misleading, a definition is in order. A mathematically perfect number is one whose factors add up to twice itself. For example, 6 is the first perfect number, since  $6 + 3 + 2 + 1 = 12$ .

Euler and Euclid have completely classified all possible even perfect numbers. Any even perfect number  $P_p$  must be of the form

$$P_p = \frac{1}{2}(M_p + 1)M_p = 2^{p-1} * (2^p - 1)$$

where  $M_p$  is the  $p$ -th Mersenne prime (see [EUL]). Euclid showed that any number of that form is an even perfect number.

However, the question

*Are there any odd perfect numbers?*

remains an open problem. For the entirety of this paper, let us assume that  $N$  is an odd perfect number, i.e.  $\sigma(N) = 2N$ , where  $\sigma$  is the sum of divisors function. We will contradict this under a variety of conditions.

We start with a result showing most odd cubes cannot be perfect numbers (see Theorem 1). Then we give a new proof of a special case of a result of Iannucci (see [IAN]) that shows that none of the even exponents in  $N$ 's prime factorization can be congruent to 4 (mod 5) if  $3|N$  (Theorem 2). We then extend that result by proving that certain sets of small primes, when taken to a large power, cannot divide an odd perfect number. This generates an upper bound on the number of small primes dividing certain odd perfect numbers (see Theorem 3-4 and Proposition 1).

Finally, we will prove that under certain conditions specific 'small' primes must be in  $N$  for it to be OP, which generates a lower bound on the number of small primes in certain odd perfect numbers (specifically, those divisible by 3) (see Theorem 5).

## *Historical Background*

Over the years, many necessary conditions have been proven in order for  $N$  to be odd perfect (hereafter OP). A very famous result, originally proved by Euler (see [EUL]), is that all odd perfect numbers  $N$  must be of the form

$$N = q^e * a_1^{2B_1} * \dots * a_n^{2B_n}$$

where  $q$  and the  $a_i$  are distinct odd primes,  $e, B_i \in \mathbb{Z}^+$ , and  $q \equiv e \equiv 1 \pmod{4}$ . The proof revolves almost entirely around considering  $\sigma(N) \pmod{4}$ . To compute  $\sigma(N)$  we will use the fact that  $\sigma$  is multiplicative, i.e.  $\sigma(\prod(p_i^{e_i})) = \prod(\sigma(p_i^{e_i}))$ , and that  $\sigma(p^k) = p^k + p^{k-1} + \dots + p + 1 = \frac{p^{k+1}-1}{p-1}$ .

To prove Euler's result, we first prove a Lemma.

**Lemma:** If  $N$  is OP, then  $N$  must have exactly one prime to an odd power in its prime factorization.

*Proof.* Suppose for the purpose of finding a contradiction that  $N$  has either two or more primes to an odd power, or none such.

We have  $N = p_1^{h_1} * p_2^{h_2} * \dots * p_n^{h_n}$  where  $p_i$  are distinct (odd) primes, and  $h_i \in \mathbb{Z}^+$ . For an arbitrary prime  $p_i$  in  $N$ 's prime factorization, note that

$$\sigma(p_i^{h_i}) := p_i^{h_i} + p_i^{h_i-1} + \dots + p_i + 1.$$

First, suppose that all the  $h_i$  are even. Then  $\sigma(p_i^{h_i})$  is a sum of  $h_i+1$  odd terms. So  $\sigma(p_i^{h_i})$  is odd. Thus  $\sigma(N) = \prod(\sigma(p_i^{h_i}))$  is odd. This contradicts the fact that  $\sigma(N) = 2N$ .

Secondly, suppose  $h_j$  and  $h_k$  are both odd, with  $j \neq k$ . Then  $\sigma(N) = \sigma(\prod(p_i^{h_i})) = \sigma(p_j^{h_j}) * \sigma(p_k^{h_k}) * \sigma(\prod(p_i^{h_i}))$  where the product is over all  $i$  with  $i \neq j, i \neq k$ . But since both  $\sigma(p_j^{h_j})$  and  $\sigma(p_k^{h_k})$  are even, this means that  $\sigma(N)$  is divisible by 4, which once again contradicts the fact that  $\sigma(N) = 2N$ .

This finishes the proof of the Lemma. □

We now prove Euler's result. This requires just a few more steps.

**Theorem :** (Euler) If  $N$  is OP, then  $N = q^e * a_1^{2B_1} * \dots * a_n^{2B_n}$ , where  $q$  and the  $a_i$  are distinct odd primes,  $e, B_i \in \mathbb{Z}^+$ , and  $q \equiv e \equiv 1 \pmod{4}$ .

*Proof.* Let  $N$  be an odd perfect number, so that  $\sigma(N) = 2N$ .

By the Lemma, we know that  $N = q^e * a_1^{2B_1} * \dots * a_n^{2B_n}$ , where  $q$  and the  $a_i$  are distinct odd primes and  $e$  is odd. It remains to show that  $q$  and  $e$  are both  $\equiv 1 \pmod{4}$ . Since

both  $q$  and  $e$  are odd, it is sufficient to show that neither can be  $\equiv 3 \pmod{4}$ .

Note that  $(q+1) \mid \sigma(q^e)$  because  $\sigma(q^e) := q^e + \dots + q + 1 = (q+1)(q^{e-1} + q^{e-3} + \dots + 1)$ .

First suppose that  $q \equiv 3 \pmod{4}$ . This means that  $(q+1) \mid \sigma(q^e) \mid \sigma(N)$ . This is a contradiction, as 4 cannot divide  $\sigma(N)$ .

Now suppose  $e \equiv 3 \pmod{4}$ . Then  $\sigma(q^e) \equiv q^e + q^{e-1} + \dots + q + 1 \equiv 1 + 1 + \dots + 1 + 1 \equiv e + 1 \equiv 0 \pmod{4}$ .

This contradicts  $2N = \sigma(N)$  and ends the proof. □

The majority of additional advances in the study of OP numbers can be divided up into three main categories.

1) Obtaining bounds for the minimal number and size of prime factors that are required to be in an OP number. The current best result is that there must be a minimum of 37 prime factors of any OP number ([IAN]), at least 8 distinct factors if  $3 \mid N$  (Hagis [HAG2], Chein [CHE]), and at least 11 distinct primes if  $N$  is not divisible by 3. (Hagis [HAG1], Kishore [KIS]).

It is also interesting to note that modern computers have proved via exhaustion that there are no odd perfect numbers smaller than  $10^{300}$  (see [BRE]).

2) Proving that no OP numbers can exist in the form  $N = q^e * a_1^{2B_1} \dots * a_n^{2B_n}$ , with all of the  $B_i$ 's in the same congruence class. McDaniel proved ([MCD2]) that having all of the  $B_i$ 's  $\equiv 1 \pmod{3}$  is sufficient for  $N$  not to be OP. Iannucci added onto this result ([IAN]) by proving that if each  $B_i$  is either  $\equiv 1 \pmod{3}$  or  $\equiv 2 \pmod{5}$  then  $N$  is not OP. The proofs of these results are quite extensive, and thus are beyond the scope of this paper. However, a new proof of a special case of Iannucci's result is provided below (Theorem 2).

Kanold proved that if all the  $B_i$ 's = 2, then  $N$  is not OP [KAN1]. Hagis and McDaniel proved that the  $B_i$ 's cannot all = 3 [HAG3]. Cohen proved that if all the  $B_i$ 's are equal to 6, 8, 11, 14, or 18, then  $N$  is not OP ([COH1]). It is also known that  $B_i$  cannot be congruent to 17 (mod 35) for all  $i$  if  $N$  is OP [MCD].

3) Proving that no OP number can exist in the form  $N = q^e * s^{2k} * a_1^{2B_1} * \dots * a_n^{2B_n}$ , where the  $B_i$ 's are all equal (usually with  $3 \mid N$ ). Even with just two primes ( $q$  and  $s$ ) allowed to vary freely these proofs are surprisingly involved. It is known that  $k$  cannot equal 2 ([BRA]) or 3 ([KAN3]). Cohen and Williams ([COH1]) proved that  $k$  cannot equal 5 or 6 when all  $B_i = 1$ .

Several results which do not fit under the umbrella of any of these three categories are also worth mentioning. Cohen ([COH2]) independently proved that if  $B_1 = 3$  and

$B_2 = 2$  with the rest of the  $B_i$ 's equal to 1, then  $N$  is not OP. Also, Kanold showed that if  $e = 5$ , and the  $B_i$ 's are any combination of 1's or 2's, then  $N$  is not OP ([KAN2]).

This historical section will end with a new proof of the nonexistence of OP numbers of the form  $N = q^e * a_1^2 * \dots * a_n^2$ . This result is originally due to Steuerwald ([STE]), but is presented here due to its relative simplicity, and as an illustration to the technique. It also serves as a good starting point for both the second and third types of analysis of OP numbers mentioned above.

We begin with the Rule of 3's (the Lemma below used in proving this Theorem 0).

**Rule of 3's:** If  $N = q^e * a_1^2 * \dots * a_n^2$  with  $q, a_i$  distinct primes,  $q, e \equiv 1 \pmod{4}$  and if  $3|N$ , then  $N$  is not OP.

*Proof.* Suppose for the purpose of contradiction that  $N$  is OP and  $3|N$ . Then since  $q \neq 3$ ,  $\sigma(3^2) = 13|N$ . If we assume that  $q$  is not in the set  $\{13, 61, 97\}$  then in turn  $\sigma(13^2) = 3 * 61|N$ ,  $\sigma(61^2) = 3 * 13 * 97|N$ , and  $\sigma(97^2) = 3 * 3169|N$ .

But here  $\sigma(13^2) \equiv \sigma(61^2) \equiv \sigma(97^2) \equiv 0 \pmod{3}$ . This gives too many 3's in  $N$ , since  $3^3 \nmid N$ .

It remains to show:  $q \neq 13, 61$ , or  $97$ .

Note that  $(q+1)|\sigma(q^e)$ , as shown in the middle of the Theorem, above.

Suppose  $q = 13$ : Then in turn  $\frac{(q+1)}{2} = 7|N$ ,  $\sigma(7^2) = 3 * 19|N$ ,  $\sigma(19^2) = 3 * 127|N$ , and  $\sigma(127^2) = 3 * 5419|N$ . This gives too many 3's.

Suppose  $q = 61$ : We already have  $3|\sigma(13^2)|N$ . Also, in turn  $\frac{(q+1)}{2} = 31|N$ ,  $\sigma(31^2) = 3 * 331|N$ , and  $\sigma(331^2) = 3 * 7 * 5233|N$ . This gives too many 3's.

Finally suppose  $q = 97$ . Then  $\frac{(q+1)}{2} = 7^2|N$ . Follow case  $q = 13$  above. This gives too many 3's, and ends the proof. □

**Theorem 0:** (Steuerwald) If  $N = q^e * a_1^2 * \dots * a_n^2$  with  $q, a_i$  distinct primes, and  $q, e \equiv 1 \pmod{4}$ , then  $N$  is not OP.

*Proof.* By the rule of 3's, it remains to show the case where 3 does not divide  $N$ . So for the purpose of finding a contradiction, suppose  $N$  is OP and not divisible by 3. This means that no  $a_i$  can be  $\equiv 0 \pmod{3}$ . Also note that  $\sigma(a_i^2) = a_i^2 + a_i + 1$ , so no  $a_i$  can be  $\equiv 1 \pmod{3}$ .

Similarly,  $q$  cannot be  $\equiv 2 \pmod{3}$ , since  $(q+1)|\sigma(q^e)|N$ . So since  $q > 3$ ,  $q$  is  $\equiv 1 \pmod{12}$ .

Let  $a_1$  be the smallest  $a_i$ .

Then  $\sigma(a_1^2) = a_1^2 + a_1 + 1 < (a_1 + 1)(a_1 + 1) < a_2^2$ , which shows that  $\sigma(a_1^2)$  is divisible by at most one  $a_i$ .

This leaves 2 cases. Either  $\sigma(a_1^2) = q^w$ , with  $1 \leq w \leq e$ , or  $\sigma(a_1^2) = q^w * a_j$  with  $0 \leq w \leq e$ .

Case 1:  $\sigma(a_1^2) = q^w * a_j$  with  $0 \leq w \leq e$ .

We already have  $a_i \equiv 2 \pmod{3}$  and  $q \equiv 1 \pmod{3}$ . This shows that  $\sigma(a_1^2)$  is  $\equiv 2^2 + 2 + 1 \equiv 1 \pmod{3}$ , and  $q^w * a_j \equiv 1^w * 2 \equiv 2 \pmod{3}$ . This is a contradiction.

Case 2:  $\sigma(a_1^2) = q^w$ , with  $1 \leq w \leq e$ .

Here  $\sigma(a_1^2) = a_1^2 + a_1 + 1 = q^w$ . This is impossible for any  $w > 1$ , by a result of Brauer (see [BRA], Lemma 1).

Thus  $w$  must be 1, and we have  $\sigma(a_1^2) = q$ .

Let  $h = \frac{q+1}{2} \equiv 1 \pmod{3}$ , so  $h$  is a product of  $a_i$ 's. If  $h$  were prime, then for some  $i$ ,  $h = a_i$ . But then  $h$  would be  $\equiv 2 \pmod{3}$ , which is false. Therefore  $h$  has at least two prime factors, the smallest of which must be  $\leq \sqrt{h}$ . Thus  $a_1 \leq \sqrt{h}$ . Thus  $q = a_1^2 + a_1 + 1 \leq h + \sqrt{h} + 1 = \frac{q+3}{2} + \sqrt{\frac{q+1}{2}}$ , so  $\frac{q-3}{2} \leq \sqrt{\frac{q+1}{2}}$ . This forces  $q \leq 7$ , which contradicts the fact that  $q \equiv 1 \pmod{12}$ . □

For an additional application of the Rule of 3's, see Theorem 5 in the Theorems section.

### *Preliminary Lemmas*

The following Lemmas are helpful for proving Theorems 1 – 5 in this paper.

**Lemma A:**  $\sigma(g^{6t}) \equiv 1 \pmod{3}$  for  $g$  prime,  $t \in \mathbb{Z}^+$ .

*Proof.* Here we will use the facts that  $g$  must be congruent to 0, 1 or 2  $\pmod{3}$ , and that  $\sigma(p^t) := p^t + p^{t-1} + \dots + p + 1$ .

Case  $g \equiv 0 \pmod{3}$ :  $\sigma(g^{6t}) \equiv 0 + 0 + \dots + 0 + 1 \equiv 1 \pmod{3}$ .

Case  $g \equiv 1 \pmod{3}$ :  $\sigma(g^{6t}) \equiv 1 + 1 + \dots + 1 + 1 \equiv 6t + 1 \equiv 1 \pmod{3}$ .

Case  $g \equiv 2 \pmod{3}$ :  $\sigma(g^{6t}) = \frac{g^{6t+1}-1}{g-1} \equiv \frac{-1^{6t+1}-1}{-1-1} \equiv \frac{-2}{-2} \equiv 1 \pmod{3}$ . □

**Lemma B:**  $\sigma(q^{3t}) \equiv 0 \pmod{3}$  for prime  $q \equiv 2 \pmod{3}$ , and  $t$  odd  $\in \mathbb{Z}^+$ .

*Proof.*  $\sigma(q^{3t}) = \frac{q^{3t+1}-1}{q-1} \equiv \frac{1-1}{2-1} \equiv 0 \pmod{3}$ . □

**Lemma C:**  $\frac{\sigma(p^k)}{p^k}$  is monotonically increasing in  $k$  and decreasing in  $p$ .

*Proof.*  $\frac{\sigma(p^k)}{p^k} = \frac{p^{k+1}-1}{(p-1)p^k} = \frac{p-\frac{1}{p^k}}{p-1}$ .

We will show that  $\frac{\sigma(p^n)}{p^n} \geq \frac{\sigma(p^m)}{p^m} \geq \frac{\sigma(q^m)}{q^m}$  if  $n \geq m$  and  $q \geq p$ .

As  $m$  increases,  $\frac{1}{p^m}$  decreases, thereby increasing the entire fraction  $\frac{\sigma(p^m)}{p^m}$ . This proves the first inequality. The second inequality follows as  $\frac{x-\frac{1}{x^m}}{x-1}$  is a decreasing function of  $x$  for  $x \geq 2$ . This exercise is left for the reader. □

**Lemma D:**  $\sigma(s^f) | \sigma(s^{f+(f+1)m})$  for all prime  $s$  and all  $m, f \in \mathbb{Z}^+$ .

*Proof.*  $\sigma(s^f) = \frac{s^{f+1}-1}{s-1}$ , and  $\sigma(s^{f+(f+1)m}) = \frac{s^{f+1+(f+1)m}-1}{s-1} = \frac{s^{(f+1)(1+m)}-1}{s-1}$ . Here the result follows after multiplication by  $s-1$ , since  $(y-1) | (y^k-1)$  for all  $y, k \in \mathbb{Z}^+$ , with  $y \neq 1$ . □

**Lemma E:** Suppose  $N = q^e * \prod a_i^{2B_i}$  is OP with all  $B_i \equiv 2 \pmod{5}$ , and  $3 | N$ . Additionally suppose that  $t_i^{e_i}$  with  $1 \leq i \leq m$  are prime powers such that  $\prod \frac{\sigma(t_i^{e_i})}{t_i^{e_i}} > 2$ , where the product is over  $i$  from 1 to  $m$ . If  $s_i^{c_i}$  are prime powers which divide  $N$  such that  $s_i \leq t_i$ , and  $c_i \geq e_i$ , then  $N$  is not OP.

*Proof.* We have  $2 = \frac{\sigma(N)}{N} > \prod \frac{\sigma(s_i^{c_i})}{s_i^{c_i}} \geq \prod \frac{\sigma(t_i^{e_i})}{t_i^{e_i}}$ , where the last inequality follows from Lemma C. By hypothesis,  $\prod \frac{\sigma(t_i^{e_i})}{t_i^{e_i}} > 2$ . This is a contradiction. □

### Theorems

We will start first with an elementary Theorem which provides necessary conditions for cubes to be OP.

**Theorem 1:** Cube Rule: If  $N$  is a cube, and  $q \equiv 1 \pmod{3}$  or  $3 \nmid N$ , then  $N$  is not an odd perfect number.

*Proof.* Suppose  $N$  is a OP and a cube, which implies  $N = m^3$  for some  $m \in \mathbb{Z}^+$ . Then

$$(1) \sigma(N) = 2N = 2(q^{3t} * p_1^{6h_1} * p_2^{6h_2} * \dots * p_k^{6h_k}), \text{ for distinct primes } p_i \text{ and } q, \text{ with } t \text{ odd.}$$

Note that the power of each odd prime factor of  $\sigma(N)$  is  $\equiv 0 \pmod{3}$ .

By Lemma A, this means that the  $\sigma(N)$  is the product of  $\sigma(q^{3t})$  times factors congruent to 1  $\pmod{3}$ . So  $\sigma(N)$  is  $\equiv \sigma(q^{3t}) \pmod{3}$  for some odd  $e$ .

First assume that  $q \equiv 1 \pmod{3}$ . We know the  $\sigma(N) \equiv 3t + 1 \equiv 1 \pmod{3}$ , which implies that the RHS of (1) must also be  $\equiv 1 \pmod{3}$ . So far this gives us

$2(q^{3t} * p_1^{6h_1} * p_2^{6h_2} * \dots * p_k^{6h_k}) \equiv 1 \pmod{3}$ . Divide through by 2 to get

$$(2) \quad q^{3t} * p_1^{6h_1} * p_2^{6h_2} * \dots * p_k^{6h_k} \equiv 2 \pmod{3}.$$

Since  $q \equiv 1 \pmod{3}$ ,  $q^{3t} \equiv 1 \pmod{3}$  as well. So by (2), at least one of the  $p_i^{6h_i}$  must be  $\equiv 2 \pmod{3}$ . That is impossible since 2 is not a square  $\pmod{3}$ .

Now we will assume that  $3 \nmid N$  (with  $q \equiv 2 \pmod{3}$ ). This implies that  $3 \nmid \sigma(q^{3t})$ , since  $\sigma(q^{3t})|N$ . This contradicts Lemma B and finishes the proof.  $\square$

**Theorem 2:** If  $3|N$ ,  $N = q^e * a_1^{2B_1} * a_2^{2B_2} * \dots * a_n^{2B_n}$  with  $q$  and  $a_i$  distinct primes,  $B_1 \equiv B_2 \equiv \dots \equiv B_n \equiv 2 \pmod{5}$ ,  $q \equiv e \equiv 1 \pmod{4}$ , and  $e, B_i \in \mathbb{Z}^+$ , then  $N$  is not an odd perfect number.

*Proof.* Suppose  $3|N$  with  $N$  OP. By Lemma D (with  $f = 4$ ),  $\sigma(a_i^4)|\sigma(a_i^{2B_i})$  for all  $i$  with  $1 \leq i \leq n$ .

In 1941, Kanold ([KAN1]) proved that the 4th power of any common factor of  $2B_i + 1$  for all  $1 \leq i \leq n$  divides  $N$ . By this result,  $5^4|N$ .

By hypothesis,  $3^4|N$ , since  $q \neq 3$ . Therefore,  $\sigma(3^4) = 11^2|N$ , or  $11^4|N$ , since  $q \neq 11$ . But this means that (by Lemma C)

$$\frac{\sigma(N)}{N} > \frac{\sigma(3^4)}{3^4} * \frac{\sigma(5^4)}{5^4} * \frac{\sigma(11^4)}{11^4} = \frac{121}{3^4} * \frac{781}{5^4} * \frac{16105}{11^4} = \frac{1521938605}{741200625} = 2.053342 > 2.$$

This contradicts  $\sigma(N) = 2N$ .  $\square$

**Theorem 3:** If  $3|N$ ,  $N = q^e * a_1^{2B_1} * a_2^{2B_2} * \dots * a_n^{2B_n}$  with  $q$  and  $a_i$  distinct primes,  $B_1 \equiv B_2 \equiv \dots \equiv B_n \equiv 38 \pmod{77}$  and  $q \equiv e \equiv 1 \pmod{4}$ ,  $B_i \in \mathbb{Z}^+$ , then  $N$  is not an odd perfect number.

*Proof.* Suppose  $N$  is OP and  $3|N$ . In 1941, Kanold ([KAN1]) proved that the 4th power of any common factor of  $2B_i + 1$  for all  $1 \leq i \leq n$  divides  $N$ . In addition, by Lemma D ( $f = 76$ ) we know that  $\sigma(a_i^{76})|\sigma(a_i^{2B_i})|N$ .

By Kanold, we know that  $7$  and  $11|N$ . By hypothesis, we know  $3|N$ . Since  $q$  is not in the set  $\{3, 7, 11, 23\}$  (as  $q \equiv 1 \pmod{4}$ ), we note that  $\sigma(3^{76}) = 23 * 1093 * 3851 * C_2|N$ , for some large pseudo-prime  $C_2$ . Most importantly, this means that  $23^4|N$ , since it is by hypothesis one of the  $a_i$ . This in turn means that by Lemma D (with  $f = 76$ ),  $(3 * 7 * 11 * 23)^{76}|N$ .

Let  $T_1 = \{3, 7, 11, 23\}$ . Then  $\frac{\sigma(N)}{N} > \frac{\prod(\sigma(t^4))}{\prod(t^4)}$  where the product is over  $t \in T_1$ . So (by Lemma C)

$$\frac{\sigma(N)}{N} \geq \frac{\sigma(3^4)}{3^4} * \frac{\sigma(7^4)}{7^4} * \frac{\sigma(11^4)}{11^4} * \frac{\sigma(23^4)}{23^4} = \frac{121}{81} * \frac{2801}{2401} * \frac{16105}{14641} * \frac{292561}{279841} = 2.004086 > 2.$$

$\square$

**Theorem 4:** If  $3|N$ ,  $N = q^e * a_1^{2B_1} * a_2^{2B_2} * \dots * a_n^{2B_n}$  with  $q$  and  $a_i$  distinct primes,  $B_1 \equiv B_2 \equiv \dots \equiv B_n \equiv \frac{(7*11*k-1)}{2} \pmod{7*11*k}$  and  $e, B_i, k \in \mathbb{Z}^+$ , and with  $q \equiv e \equiv 1 \pmod{4}$ , then  $N$  is not an odd perfect number.

*Proof.* Suppose  $N$  is OP,  $3|N$  and all  $B_i \equiv \frac{77k-1}{2} \pmod{77k}$ .

Then all  $B_i \equiv 38 \pmod{77}$  because

$$2 * B_i \text{ (for all } i \text{ from 1 to } n) \equiv 2 * \frac{77k-1}{2} \equiv 77k - 1 \equiv 76 \pmod{77}.$$

Therefore the result follows from Theorem 3. □

This method gives a contradiction whenever a product  $p_1^4 * p_2^4 * \dots * p_k^4$  which divides  $N$  generates a value of  $\frac{\sigma(N)}{N} > 2$ . Such products include:

- (3)  $(3 * 5 * 13)^4$ ,
- (4)  $(3 * 5 * 17 * 113)^4$ ,
- (5)  $(3 * 5 * 19 * 67)^4$ ,
- (6)  $(3 * 7 * 13 * 17)^4$ ,
- (7)  $(3 * 7 * 11 * 23)^4$ ,
- (8)  $(3 * 11 * 13 * 17 * 19 * 563)^4$ , and
- (9)  $(5 * 7 * 11 * 13 * 19 * 37)^4$ .

Many more examples can be obtained from (3) – (9) through application of Lemma C. For example,  $(3 * 5 * 7)^4$  and  $(3 * 5 * 11)^4$  follow from (3), and  $(3 * 5 * 17 * 19)^4$  follows from (4). Some more examples are provided in the proposition below.

The following proposition shows that no odd perfect number can contain five small primes that include 3 and 5 (with all primes to the 4th or higher power).

**Proposition:** Small Primes Rule: If  $3^4, 5^4, p_1^4, p_2^4$ , and  $p_3^4|N$  with  $p_i$  distinct primes between 6 and 46, then  $N$  is not odd perfect.

*Proof.* As in the proof of Theorem 4,

$$\frac{\sigma(N)}{N} > \frac{\sigma(3^4)}{3^4} * \frac{\sigma(5^4)}{5^4} * \frac{\sigma(43^4)}{43^4} * \frac{\sigma(41^4)}{41^4} * \frac{\sigma(37^4)}{37^4} = \frac{121}{81} * \frac{781}{625} * \frac{3500201}{3418801} * \frac{2896405}{2825761} * \frac{1926221}{1874161} = 2.013323 > 2,$$

so  $N$  cannot be OP. This follows by Lemma C. □

In summary, we have the following generalization of Theorem 4:

If  $N = q^e * a_1^{2B_1} * a_2^{2B_2} * \dots * a_n^{2B_n}$  with  $q$  and  $a_i$  distinct primes,  $B_1 \equiv B_2 \equiv \dots \equiv B_n \equiv \frac{1}{2} * (p_1 * p_2 * \dots * p_j * k - 1) \pmod{p_1 * p_2 * \dots * p_j * k}$  and  $e, B_i, k \in \mathbb{Z}^+$   $q \equiv e \equiv 1 \pmod{4}$



4), and  $\prod(p_i)$  is a product of primes such as those in (3) – (9) s.t.  $\frac{\sigma(\prod p_i^4)}{\prod p_i^4} > 2$ , then  $N$  is not an odd perfect number.

This paper will conclude with a strengthening of 'The Rule of 3's' (originally proved in the historical section).

**Theorem 5:** If  $3|N$ , and if the power of the prime 3 in  $N$  is exactly 2, and if the powers of 13, 61, and 97 (if they appear) are  $\equiv 2 \pmod{3}$ , then  $N$  is not an odd perfect number.

*Proof.* Suppose for the purpose of finding a contradiction that  $N$  is OP and  $3|N$ . By hypothesis, this means  $3^2||N$ , and therefore  $\sigma(3^2) = 13|N$ .

Note that  $N = q^e * a_1^{2B_1} * a_2^{2B_2} * \dots * a_n^{2B_n}$  with  $q$  and  $a_i$  distinct primes,  $q \equiv e \equiv 1 \pmod{4}$ , and all  $B_i \in \mathbb{Z}^+$ .

In the following lines we will be using Lemma D (with  $f = 2$ ).

Note that  $\sigma(13^2)|\sigma(13^{2B_1})|N$ , where  $13^{2B_1}||N$  (for some  $B_1 \in \mathbb{Z}^+$ ). This implies that  $\sigma(13^2) = 3 * 61|N$ .

Similarly,  $\sigma(61^2)|\sigma(61^{2B_2})|N$ , where  $61^{2B_2}||N$  (for some  $B_2 \in \mathbb{Z}^+$ ). This means that  $\sigma(61^2) = 3 * 13 * 97|N$ .

Finally,  $\sigma(97^2)|\sigma(97^{2B_3})|N$  with  $97^{2B_3}||N$  (for some  $B_3 \in \mathbb{Z}^+$ ). This implies that  $\sigma(97^2) = 3 * 3169|N$ .

This means that  $\sigma(97^{2B_3}) \equiv \sigma(61^{2B_2}) \equiv \sigma(13^{2B_1}) \equiv 0 \pmod{3}$ , and thus  $3^3|N$ . This is a contradiction.  $\square$

This result provides a lower bound when the powers of 3, 13, 61, and 97 (in an integer  $N$ ) are exactly 2. In this case,  $N$  cannot be odd perfect by Theorem 5. So if  $N$  is OP with  $3|N$  either:

A)  $3^2||N$  and the power of one of 13, 61, and 97 must be 3 or higher, or

B)  $3^4|N$  (since  $q$  cannot be 3 since  $q \equiv 1 \pmod{4}$ , and  $q$  is the only prime that can be taken to an odd power).

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