Non-Commutative Subharmonic and Harmonic Polynomials

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1 Introduction

1.1 Non-Commutative Algebra in Engineering

Inequalities, involving polynomials in matrices and their inverses, and associated optimization problems have become very important in engineering. When the said polynomials are matrix convex, interior point methods apply directly. In the last few years, the approaches that have been proposed in the field of optimization and control theory based on linear matrix inequalities and semidefinite programming have become very important and promising, since the same framework can be used for a large set of problems. Matrix inequalities provide a nice setup for many engineering and related problems, and if they are convex the optimization problem is well behaved and interior point methods provide efficient algorithms which are effective on moderate sized problems. Unfortunately, the class of convex polynomials in matrix variables is very small - it has been proven, in fact, that they are all of degree two or less [HS04]. It is in our interest, then, to discover conditions similar to convexity, but not as restrictive, and to characterize these classes thoroughly.

2 NC Polynomials

Definition 2.1 *Non-Commutative Monomials*

A non-commutative monomial m of degree d on the indexed free variables $x_1, \ldots, x_g$ is a product $x_{a_1}x_{a_2}\cdots x_{a_d}$ of these variables, corresponding to a unique sequence of a_i. A convenient convention is to write $m = x^w$, where w is the d-tuple $\langle a_1, \ldots, a_d \rangle$. We treat w as a word on the "alphabet" $[g] = \{1, \ldots, g\}$ and m, consequently, as a word on $x = \{x_1, \ldots, x_g\}$. Some word-conventions:

$\lvert w\rvert = d$	the length of w
$(w)_i = a_i$	the i^{th} letter of w
$w^T = \langle a_d, \ldots, a_1 \rangle$	the transpose of w
$\phi = \langle\rangle$	the empty word (word of length zero)

Finally, for any two words $w_1 = \langle a_1, \ldots, a_{d_1} \rangle$ and $w_2 = \langle b_1, \ldots, b_{d_2} \rangle$, we have $w_1w_2 = \langle a_1, \ldots, a_{d_1}, b_1, \ldots, b_{d_2} \rangle$, called "$w_1$ concatenate w_2".

Definition 2.2 *Non-Commutative Polynomials*

A non-commutative polynomial $p(x) = p(x_1, \ldots, x_g)$ is the sum of non-commutative monomials (on g variables) m, to which we associate scalar coefficients $A_m \in \mathbb{R}$; m is a word ranging over a given set $\mathcal{M}_p$. The degree of p is equal to the length of the longest word in $\mathcal{M}_p$.

$$p(x) = \sum_{m \in \mathcal{M}_p} A_m \, m. \tag{1}$$

Example 2.1 *A non-commutative polynomial*

$$p(x) = p(x_1, x_2) = x_1^2 x_2 x_1 + x_1 x_2 x_1^2 + x_1 x_2 - x_2 x_1 + 7$$

(on commutative variables, this would be equivalent to $2x_1^2 x_2 + 7$).

In this example,

$$\mathcal{M}_p = \{\langle x_1, x_1, x_2, x_1\rangle, \langle x_1, x_2, x_1, x_1\rangle, \langle x_1, x_2\rangle, \langle x_2, x_1\rangle, \phi\}.$$

2.1 NC-Symmetric Polynomials

The transpose of a polynomial p, denoted p^T, has the following properties:

(1) $(p^T)^T = p$
(2) $(p_1 + p_2)^T = p_1^T + p_2^T$
(3) $(\alpha p)^T = \alpha p^T \qquad (\alpha \in \mathbb{R})$
(4) $(p_1 p_2)^T = p_2^T p_1^T$.

In this paper, we shall consider only polynomials in symmetric variables. That is, we consider variables x_i where $x_i^T = x_i$. Transposition of a polynomial in such variables amounts to reversing the "spelling" of each of its terms.

We know that $\quad x_{a_1}^T = x_{a_1}$

Assuming that $\quad (x_{a_1} \ldots x_{a_{d-1}})^T = x_{a_{d-1}} \ldots x_{a_1}$, we have

$$(x_{a_1} \ldots x_{a_d})^T = x_{a_d}^T (x_{a_1} \ldots x_{a_{d-1}})^T = x_{a_d}(x_{a_{d-1}} \ldots x_{a_1}) = x_{a_d} \ldots x_{a_1}.$$

Returning to our indexed-word notation, we find the identity

$$(x^w)^T = x^{w^T}. \tag{3}$$

Symmetric (or self-adjoint) polynomials are those that are equal to their transposes. We will consider this class of functions in our investigation.

2.2 Matrix Positivity

Definition 2.3 *Matrix Positive Symmetric Polynomials*

A symmetric polynomial $p(x) = p(x_1, \ldots, x_g)$ is considered matrix-positive over a domain D of square symmetric matrices $X = (X_1, \ldots, X_g)$ iff, for any $X \in D$, we have that $p(X)$ is a positive-semidefinite matrix. Note that if $p(x) = 1$, then $p(X) = I_D$. For instance:

Example 2.2 $p(x) = x_1^2 + \ldots + x_g^2$ *is matrix-positive.*

Substituting in symmetric matrices X_i for x_i, we have for any vector v

$$\begin{aligned} v^T(X_1^2 + \ldots + X_g^2)v &= v^T X_1^2 v + \ldots + v^T X_g^2 v \\ &= (X_1 v)^T (X_1 v) + \ldots + (X_g v)^T (X_g v) \\ &= \|X_1 v\|^2 + \ldots \|X_g v\|^2 \ \geq \ 0, \end{aligned}$$

that is, $p(X)$ is positive-semidefinite, for all X. By definition $p(x)$ is matrix-positive.

2.3 Non-Commutative Differentiation

For our non-commutative purposes, we take directional derivatives in x_i with regard to an indeterminate direction parameter h [CHSY03].

$$\mathrm{DirD}[p(x_1, \ldots, x_g), x_i, h] := \frac{d}{dt}[p(x_1, \ldots, (x_i + th), \ldots, x_g)]|_{t=0}. \tag{4}$$

We say that this is the directional deriative of $p(x) = p(x_1, \ldots, x_g)$ on x_i in the direction h.

Example 2.3 *The directional derivative*

$$\begin{aligned}\text{DirD}[x_1^2x_2, x_1, h] &= \tfrac{d}{dt}[(x_1+th)^2x_2]|_{t=0} \\ &= \tfrac{d}{dt}[x_1^2x_2 + thx_1x_2 + tx_1hx_2 + t^2h^2x_2]|_{t=0} \\ &= [hx_1x_2 + x_1hx_2 + 2th^2x_2]|_{t=0} \\ &= hx_1x_2 + x_1hx_2.\end{aligned}$$

As this example shows, the directional derivative of p on x_i in the direction h is the sum of the terms produced by replacing one instance of x_i with h.

Lemma 2.1 *Linearity of the directional derivative of NC polynomials*

$$\text{DirD}[a\,p(x) + b\,q(x), x_i, h] = a\ \text{DirD}[p(x), x_i, h] + b\ \text{DirD}[q(x), x_i, h].$$

Proof:

$$\begin{aligned}&\text{DirD}[a\,p(x) + b\,q(x), x_i, h] \\ &\qquad= \frac{d}{dt}[(a\,p(x) + b\,q(x))(x_1, \ldots, (x_i+th), \ldots, x_g)]|_{t=0} \\ &\qquad= \frac{d}{dt}[a\,p(x_1, \ldots, (x_i+th), \ldots, x_g) + b\,q(x_1, \ldots, (x_i+th), \ldots, x_g)]|_{t=0}.\end{aligned}$$

By linearity of the $\frac{d}{dt}$ operator,

$$\begin{aligned}&= (a\frac{d}{dt}[p(x_1, \ldots, (x_i+th), \ldots, x_g)] + b\frac{d}{dt}[q(x_1, \ldots, (x_i+th), \ldots, x_g)])|_{t=0} \\ &= a\frac{d}{dt}[a\,p(x_1, \ldots, (x_i+th), \ldots, x_g)]|_{t=0} + b\frac{d}{dt}[b\,q(x_1, \ldots, (x_i+th), \ldots, x_g)]|_{t=0} \\ &= a\,\text{DirD}[p(x), x_i, h] + b\,\text{DirD}[q(x), x_i, h].\end{aligned}$$

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2.4 Non-Commutative Convexity

The non-commutative Hessian is defined as:

$$\text{NCHessian}[p(x_1, \ldots, x_g), \{x_1, \eta_1\}, \ldots, \{x_g, \eta_g\}] := \frac{d^2}{dt^2}[p(x_1+t\eta_1, \ldots, x_g+t_g\eta_g)]|_{t=0}.$$

Note that this is composed of several independent direction parameters, η_i and that if p is a polynomial, then its Hessian is a polynomial in x and η which is homogeneous of degree 2 in η. *Such notation is unique to this subsection.* A non-commutative polynomial is considered **convex** wherever its Hessian is matrix-positive. Equivalently,

Definition 2.4 *Geometrical NC Symmetric Convexity*

A polynomial $p(x) = p(x_1, \ldots, x_d)$ is convex over a convex domain D of ordered d-tuples $X = (X_1, \ldots, X_d)$ of square matrices if and only if, for every X and Y in D,

$$\frac{1}{2}\big(p(X) + p(Y)\big) - p\left(\frac{X+Y}{2}\right)$$

is positive-semidefinite. It is proved in [HM98] that convexity is equivalent to geometric convexity. A crucial fact regarding these polynomials (proven by Helton, et. al [HS04]) is that they are all of degree two or less.

The commutative analog of this "directional" Hessian is the quadratic function

$$H\big(p(x)\big)\begin{pmatrix}\eta_1\\ \vdots\\ \eta_g\end{pmatrix}\cdot\begin{pmatrix}\eta_1\\ \vdots\\ \eta_g\end{pmatrix} \tag{7}$$

where $H\big(p(x)\big)$ is the Hessian matrix:

$$\begin{pmatrix}\partial_{x_1x_1}p(x) & \cdots & \partial_{x_1x_g}p(x)\\ \vdots & \ddots & \vdots\\ \partial_{x_gx_1}p(x) & \cdots & \partial_{x_gx_g}p(x)\end{pmatrix}. \tag{8}$$

If this Hessian is positive- (resp. negative-) semidefinite at some critical point $(x_1, \ldots, x_g)$, then f has a local minimum (resp. maximum) at that point, and f is said to be "concave" (resp. "convex") there. This corresponds, somewhat, to our intuitive physical notions of convexity.

2.5 Non-Commutative Laplacian and Subharmonicity

The condition of subharmonicity is much less restrictive than that of convexity. It is determined by a polynomial's non-commutative Laplacian, which we define as:

$$\begin{aligned}\mathrm{Lap}[p(x_1, \ldots, x_g), h] &:= \sum_{i=1}^{g} \mathrm{DirD}[\mathrm{DirD}[p(x), \{x_i, h\}], \{x_i, h\}] &(9)\\ &= \sum_{i=1}^{g} \frac{d^2}{dt^2}[p(x_1, \ldots, (x_i + th), \ldots, x_g)]|_{t=0}. &(10)\end{aligned}$$

Note that (9) defines Lap as a finite sum of compositions of linear functions (DirD); thus, it too must be linear.

Analogous to the directional Laplacian, in commutative variables, is:

$$\Delta\big(p(x)\big)h^2 \tag{11}$$

where $\Delta\big(p(x)\big)$ is the standard Laplacian, namely:

$$\Delta\big(p(x)\big) := \sum_{i=1}^{g} \partial_{x_i x_i} p(x). \tag{12}$$

A polynomial is considered "harmonic" if its Laplacian is zero, and "subharmonic" if its Laplacian is matrix-positive ("properly subharmonic" is used to describe a polynomial which is subharmonic but not harmonic - that is, having a nonzero, matrix-positive Laplacian).

3 Main Theorem (Two Variables)

For our special homogeneous polynomials on two variables, define

$$\gamma := x_1 + i\, x_2 \tag{13}$$

where i is the imaginary number.

Theorem 1 *For $d > 2$, the homogeneous NC symmetric polynomials in two symmetric variables which are*

(1) harmonic of degree d include the linear combinations of

$$\mathrm{Re}(\gamma^d) \quad \textit{and} \quad \mathrm{Im}(\gamma^d),$$

(2) subharmonic of degree $2d$, include the linear combinations:

$$\begin{aligned} &c_0[\mathrm{Re}(\gamma^d)]^2 \,+\, c_1\,\mathrm{Re}(\gamma^{2d}) \,+\, c_2\,\mathrm{Im}(\gamma^{2d}) \\ &\quad = c_0[\mathrm{Im}(\gamma^d)]^2 \,+\, (c_0 + c_1)\,\mathrm{Re}(\gamma^{2d}) \,+\, c_2\,\mathrm{Im}(\gamma^{2d}) \end{aligned} \tag{15}$$

where $c_0 \geq 0$,

(3) subharmonic of odd degree do not exist.

All NC symmetric harmonic polynomials of degree 2, and the subharmonic polynomials of degree 4 or less are listed:

Degree 2:

Spanning Polynomial	*Characteristics*
$A_1\, x_1^2 + A_2\, x_2^2$ $+A_3\,(x_1x_2 + x_2x_1)$	*Region of Subharmonicity* $A_1 + A_2 > 0$ *Region of Harmonicity* $A_1 + A_2 = 0$

Degree 3:

Spanning Polynomial	*Characteristics*
$A_1\,(x_1^3 - x_1x_2^2 - x_2x_1^2)$ $+A_2\, x_2x_1x_2 + A_3\, x_1x_2x_1$ $+A_4\,(x_2^3 - x_1^2x_2 - x_2x_1^2)$	*Region of Subharmonicity* $(A_1 + A_2)x_1 + (A_3 + A_4)x_2 > 0$ *Region of Harmonicity* $A_1 + A_2 = A_3 + A_4 = 0$

[1]

Degree 4:

Spanning Polynomial	*Characteristics*
$A_1\,(x_1^4 - x_1^2x_2^2 - x_2^2x_1^2 + x_2^4)$ $+A_2\, x_1x_2^2x_1 + A_3\, x_2x_1^2x_2$ $+A_4\,(x_1x_2x_1x_2 + x_2x_1x_2x_1)$ $+A_5\,(x_2^2x_1x_2 - x_1^3x_2$ $-x_2x_1^3 + x_2x_1x_2^2)$ $+A_6\,(x_1^2x_2x_1 + x_1x_2x_1^2$ $-x_1x_2^3 - x_2^3x_1)$	*Region of Subharmonicity* $(A_1 + A_2)(A_1 + A_3) > (A_1 + A_4)^2 + (A_5 + A_6)^2$ $A_1 + A_2 > 0$ *or, equivalently,* $A_1 + A_3 > 0$ *Region of Harmonicity* $A_1 + A_2 = A_1 + A_3 = A_1 + A_4 = 0$ $A_5 = A_6 = 0$

Conjecture 3.1 *The subharmonic and harmonic polnomials are none other than those described above.*

[1]In the degree 3 case, note that the subharmonic region depends on x_1 and x_2. Therefore, there is no polynomial of degree three which is subharmonic over all values of x_1 and x_2

This is a result we believe we have proven but there was not time to write the proof out carefully in this thesis.

4 Existence Proofs

4.1 Product Rules for Derivatives

4.1.1 Product Rule for DirD

Lemma 4.1 *The product rule for the directional derivative of NC polynomials is*

$$\mathrm{DirD}[p_1 p_2, x_i, h] = \mathrm{DirD}[p_1, x_i, h] p_2 \; + \; p_1 \, \mathrm{DirD}[p_2, x_i, h].$$

Proof: The directional derivative $\mathrm{DirD}[m, x_i, h]$ of a product $m = m_1 m_2$ of non-commutative monomials m_1 and m_2 is the sum of terms produced by replacing one instance of x_i in m by h. This sum can be divided into two parts:

μ_1, the sum of terms whose h lie in the first $|m_1|$ letters,

$$\text{i.e.} \quad \mathrm{DirD}[m_1, x_i, h] m_2$$

μ_2, the sum of terms whose h lie in the last $|m_2|$ letters,

$$\text{i.e.} \quad m_1 \, \mathrm{DirD}[m_2, x_i, h].$$

Therefore

$$\mathrm{DirD}[m_1 m_2, x_i, h] = \mathrm{DirD}[m_1, x_i, h] m_2 + m_1 \, \mathrm{DirD}[m_2, x_i, h].$$

Noting the linearity of the directional derivative, we can extend this product rule to the product of any two non-commutative polynomials p_1 and p_2.

$$\begin{aligned}
\mathrm{DirD}[p_1 p_2, x_i, h] &= \mathrm{DirD}\Big[\Big(\sum_{m_1 \in \mathcal{M}_{p_1}} A_{m_1} m_1\Big)\Big(\sum_{m_2 \in \mathcal{M}_{p_2}} A_{m_2} m_2\Big), x_i, h\Big] \\
&= \sum_{m_1 \in \mathcal{M}_{p_1}} \sum_{m_2 \in \mathcal{M}_{p_2}} A_{m_1} A_{m_2} \, \mathrm{DirD}[m_1 m_2, x_i, h] \\
&= \sum_{m_1 \in \mathcal{M}_{p_1}} \sum_{m_2 \in \mathcal{M}_{p_2}} A_{m_1} A_{m_2} \, \mathrm{DirD}[m_1, x_i, h] m_2 + A_{m_1} A_{m_2} \, m_1 \, \mathrm{DirD}[m_2 x_i, h] \\
&= \mathrm{DirD}\Big[\sum_{m_1 \in \mathcal{M}_{p_1}} A_{m_1} m_1, x_i, h\Big] \sum_{m_2 \in \mathcal{M}_{p_2}} A_{m_2} m_2 + \sum_{m_1 \in \mathcal{M}_{p_1}} A_{m_1} m_1 \, \mathrm{DirD}\Big[\sum_{m_2 \in \mathcal{M}_{p_2}} A_{m_2} m_2, x_i, h\Big] \\
&= \mathrm{DirD}[p_1, x_i, h] p_2 + p_1 \, \mathrm{DirD}[p_2, x_i, h]. \qquad (19)
\end{aligned}$$

4.1.2 The Laplacian of a Product

Lemma 4.2 *The product rule for the Laplacian of NC polynomials is*

$$\mathrm{Lap}[p_1\, p_2, h] = \mathrm{Lap}[p_1, h]\, p_2 + p_1\, \mathrm{Lap}[p_2, h] + 2\sum_{i=1}^{g}\big(\mathrm{DirD}[p_1, x_i, h]\, \mathrm{DirD}[p_2, x_i, h]\big).$$

Proof:

$$\begin{aligned}
\mathrm{Lap}[p_1\, p_2, h] &= \sum_{i=1}^{g} \mathrm{DirD}[\mathrm{DirD}[p_1\, p_2, x_i, h], x_i, h] \\
&= \sum_{i=1}^{g} \mathrm{DirD}\, [p_1\ \mathrm{DirD}[p_2, x_i, h] + \mathrm{DirD}[p_1, x_i, h]\, p_2,\ x_i, h]\big) \\
&= \sum_{i=1}^{g} \big(p_1\ \mathrm{DirD}[\mathrm{DirD}[p_2, x_i, h], x_i, h] + \mathrm{DirD}[\mathrm{DirD}[p_1, x_i, h], x_i, h]\, p_2 \\
&\qquad + 2\, \mathrm{DirD}[p_1, x_i, h], \mathrm{DirD}[p_2, x_i, h]\big) \\
&= \mathrm{Lap}[p_1, h]\, p_2 + p_1\, \mathrm{Lap}[p_2, h] + 2\sum_{i=1}^{g}\big(\mathrm{DirD}[p_1, x_i, h]\, \mathrm{DirD}[p_2, x_i, h]\big).
\end{aligned}$$

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4.2 Formulas Involving Derivatives of γ^d

Recall

$$\gamma := x_1 + i x_2.$$

Lemma 4.3 *The derivatives of of γ^d exhibit the following symmetries.*

$$\mathrm{DirD}[\mathrm{Re}(\gamma^d), x_1, h] = \mathrm{DirD}[\mathrm{Im}(\gamma^d), x_2, h]$$

and

$$\mathrm{DirD}[\mathrm{Re}(\gamma^d), x_2, h] = -\,\mathrm{DirD}[\mathrm{Im}(\gamma^d), x_1, h].$$

Proof:

It is easily seen that

$$\begin{aligned}
\mathrm{DirD}[\mathrm{Re}(\gamma), x_1, h] &= \mathrm{DirD}[\mathrm{Im}(\gamma), x_2, h] = h \\
\mathrm{DirD}[\mathrm{Im}(\gamma), x_1, h] &= -\,\mathrm{DirD}[\mathrm{Re}(\gamma), x_2, h] = 0.
\end{aligned}$$

Assume that

$$\begin{aligned}\mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_1, h] &= \mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_2, h]\\ \mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_1, 1] &= -\,\mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_2, h].\end{aligned}$$

Then

$$\begin{aligned}\mathrm{DirD}[\,\mathrm{Re}(\gamma^{d}), x_1, h] &= \mathrm{DirD}[x_1\mathrm{Re}(\gamma^{d-1}) - x_2\mathrm{Im}(\gamma^{d-1}), x_1, h]\\ &= x_1\ \mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_1, h] + h\,\mathrm{Re}(\gamma^{d-1}) - x_2\ \mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_1, h]\end{aligned}$$

$$\begin{aligned}\mathrm{DirD}[\mathrm{Im}(\gamma^{d}), x_2, h] &= \mathrm{DirD}[x_1\mathrm{Im}(\gamma^{d-1}) + x_2\mathrm{Re}(\gamma^{d-1}), x_2, h]\\ &= x_1\ \mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_2, h] + x_2\ \mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_2, h] + h\,\mathrm{Re}(\gamma^{d-1}),\end{aligned}$$

so

$$\mathrm{DirD}[\mathrm{Re}(\gamma^{d}), x_1, h] = \mathrm{DirD}[\mathrm{Im}(\gamma^{d}), x_2, h] \tag{23}$$

which satisfies the first half of our inductive hypothesis. For the next half compute

$$\begin{aligned}\mathrm{DirD}[\,\mathrm{Re}(\gamma^{d}), x_2, h] &= \mathrm{DirD}[x_1\mathrm{Re}(\gamma^{d-1}) - x_2\mathrm{Im}(\gamma^{d-1}), x_2, h]\\ &= x_1\ \mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_2, h] - x_2\ \mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_2, h] - h\,\mathrm{Im}(\gamma^{d-1})\end{aligned}$$

$$\begin{aligned}\mathrm{DirD}[\mathrm{Im}(\gamma^{d}), x_1, h] &= \mathrm{DirD}[x_1\mathrm{Im}(\gamma^{d-1}) + x_2\mathrm{Re}(\gamma^{d-1}), x_1, h]\\ &= x_1\ \mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_1, h] + h\,\mathrm{Im}(\gamma^{d-1}) + x_2\ \mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_2, h],\end{aligned}$$

so

$$\mathrm{DirD}[\mathrm{Re}(\gamma^{d}), x_2, h] = -\,\mathrm{DirD}[\mathrm{Im}(\gamma^{d}), x_1, h]. \tag{24}$$

■

Also note that $\gamma = \gamma^T$ and therefore that $\gamma^d = (\gamma^d)^T$. So

$$\big(\mathrm{Re}(\gamma^d)\big)^T = \mathrm{Re}((\gamma^d)^T) = \mathrm{Re}(\gamma^d) \quad \text{and} \quad \big(\mathrm{Im}(\gamma^d)\big)^T = \mathrm{Im}((\gamma^d)^T) = \mathrm{Im}(\gamma^d).$$

4.3 Harmonics *degree* > 2: Proof of Theorem 1 part (1)

Since no double-substitutions can be made on words of length 1,

$$\mathrm{Lap}[\mathrm{Re}(\gamma), h] = \mathrm{Lap}[x_1, h] = 0 \quad \text{and} \quad \mathrm{Lap}[\mathrm{Im}(\gamma), h] = \mathrm{Lap}[x_2, h] = 0. \tag{26}$$

Now, assume that

$$\mathrm{Lap}[\mathrm{Re}(\gamma^{d-1}), h] = \mathrm{Lap}[\mathrm{Im}(\gamma^{d-1}), h] = 0. \tag{27}$$

Pushing ahead,

$$\mathrm{Re}(\gamma^{d}) = \mathrm{Re}((x_1 + ix_2)\,\gamma^{d-1}) = x_1\mathrm{Re}(\gamma^{d-1}) - x_2\mathrm{Im}(\gamma^{d-1}) \tag{28}$$

$$\mathrm{Im}(\gamma^{d}) = \mathrm{Im}((x_1 + ix_2)\,\gamma^{d-1}) = x_1\mathrm{Im}(\gamma^{d-1}) + x_2\mathrm{Re}(\gamma^{d-1}). \tag{29}$$

Applying our product rule to (28):

$$\begin{aligned}
\mathrm{Lap}[\mathrm{Re}(\gamma^{d}), h] &= \mathrm{Lap}[x_1\mathrm{Re}(\gamma^{d-1}), h] - \mathrm{Lap}[x_2\mathrm{Im}(\gamma^{d-1}), h] \\
&= x_1\,\mathrm{Lap}[\mathrm{Re}(\gamma^{d-1}), h] + \mathrm{Lap}[x_1, h]\mathrm{Re}(\gamma^{d-1}) \\
&\quad + 2\,\mathrm{DirD}[x_1, x_1, h]\,\mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_1, h] \\
&\quad + 2\,\mathrm{DirD}[x_1, x_2, h]\,\mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_2, h] \\
&\quad - x_2\,\mathrm{Lap}[\mathrm{Im}(\gamma^{d-1}), h] - \mathrm{Lap}[x_2, h]\mathrm{Im}(\gamma^{d-1}) \\
&\quad - 2\,\mathrm{DirD}[x_2, x_1, h]\,\mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_1, h] \\
&\quad - 2\,\mathrm{DirD}[x_2, x_2, h]\,\mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_2, h].
\end{aligned}$$

Use (26) and (27) to obtain that the *Lap*[] terms are 0, and that "cross partials are 0" to get

$$\mathrm{Lap}[\mathrm{Re}(\gamma^{d}), h] = h\,\mathrm{DirD}[\mathrm{Re}(\gamma^{d-1}), x_1, h] - h\,\mathrm{DirD}[\mathrm{Im}(\gamma^{d-1}), x_2, h].$$

By symmetry Lemma 4.3, this means

$$\mathrm{Re}(\mathrm{Lap}[\gamma^{d}, h]) = \mathrm{Lap}[\mathrm{Re}(\gamma^{d}), h] = 0. \tag{30}$$

By a similar argument, applying the product rule to (29),

$$\mathrm{Im}(\mathrm{Lap}[\gamma^{d}, h]) = \mathrm{Lap}[\mathrm{Im}(\gamma^{d}), h] = 0. \tag{31}$$

Harmonic Polynomials are further discussed in [McA04].

4.4 Subharmonics *degree* > 4: Proof of Theorem 1 (2)(3)

matrix-positive for all d.

By the product rule for the Laplacian

$$\begin{aligned}\operatorname{Lap}[\big(\operatorname{Re}(\gamma^d)\big)^2, h] &= \operatorname{Lap}[\operatorname{Re}(\gamma^d), h]\operatorname{Re}(\gamma^d) + \operatorname{Re}(\gamma^d)\operatorname{Lap}[\operatorname{Re}(\gamma^d), h] \\ &\quad + 2\big((\operatorname{DirD}[\operatorname{Re}(\gamma^d), x_1, h])^2 + (\operatorname{DirD}[\operatorname{Re}(\gamma^d), x_2, h])^2\big) \\ &= 2\big((\operatorname{DirD}[\operatorname{Re}(\gamma^d), x_1, h])^2 + (\operatorname{DirD}[\operatorname{Re}(\gamma^d), x_2, h])^2\big)\end{aligned}$$

which is a sum of squares (as $\operatorname{Re}(\gamma^d)$ is symmetric). Thus we have shown $\big(\operatorname{Re}(\gamma^d)\big)^2$ is subharmonic. The same argument applies to $\big(\operatorname{Im}(\gamma^d)\big)^2$.

By Helton's [Hel02], we know that a matrix positive polynomial *must* be expressible as a sum of squares, and therefore must have even degree. Since the directional Laplacian preserves degree, subharmonic polynomials, having matrix-positive Laplacians, must have even degree.

Now we prove the formula (15) relating subharmonics. We use from the elementary result

$$\begin{aligned}\gamma^{2d} &= (\operatorname{Re}(\gamma^d) + i\operatorname{Im}(\gamma^d))^2 \\ &= (\operatorname{Re}(\gamma^d))^2 - (\operatorname{Im}(\gamma^d))^2 + i(\operatorname{Re}(\gamma^d)\operatorname{Im}(\gamma^d) + \operatorname{Im}(\gamma^d)\operatorname{Re}(\gamma^d)).\end{aligned}$$

Therefore

$$\operatorname{Re}(\gamma^{2d}) = \big(\operatorname{Re}(\gamma^d)\big)^2 - \big(\operatorname{Im}(\gamma^d)\big)^2.$$

So

$$\begin{aligned}&c_0\left[\operatorname{Re}(\gamma^d)\right]^2 + c_1\operatorname{Re}(\gamma^{2d}) + c_2\operatorname{Im}(\gamma^{2d}) \\ &\qquad = c_0\left[\operatorname{Im}(\gamma^d)\right]^2 + (c_0 + c_1)(\operatorname{Re}(\gamma^{2d})) + c_2\operatorname{Im}(\gamma^{2d}).\end{aligned}$$

4.5 Subharmonics of Odd Degree

To see that there are no subharmonics of odd degree note that the Laplacian of an odd degree polynomial is itself an odd degree polynomial. However, by [Hel02] this polynomial being positive makes it a sum of squares. Any sum of squares has even degree, thereby producing a contradiction.

5 Classification when Degree is Four or Less

5.1 The Matrix Representation

Each polynomial quadratic in the variables $x_1, \ldots, x_g$ can be expressed as a product $q^T P q$, where P is independent of all x_i. We shall call P the "middle matrix” and q the “border vector” for this representation.

5.1.1 Commutative Coefficients

Consider the generalized quadratic polynomial

$$Q(x) = \sum_{i=1}^{g}\sum_{j=1}^{g} A_{ij}\, x_i\, x_j$$

with scalar coefficients A_{ij}. Define vector $q = q(x)$ and matrix P such that

$$p_{ij} = A_{ij} \quad \text{and} \quad q = (x_1, \ldots, x_g), \tag{34}$$

then $\quad q^T P q = \sum_{i=1}^{g} x_i \sum_{j=1}^{g} P_{ij}\, x_j = \sum_{i=1}^{g}\sum_{j=1}^{g} x_i\, A_{ij}\, x_j = \sum_{i=1}^{g}\sum_{j=1}^{g} A_{ij}\, x_i\, x_j = Q(x).$

It is easy to see that when $Q(x)$ is symmetric (i.e. when the coefficient of $x_i\, x_j$ equals that of $x_j\, x_i$) P must also be symmetric:

$$p_{ij} = A_{ij} = A_{ji} = p_{ji}.$$

Example 5.1 *The Matrix Representation of the general quadratic, $g = 2$*

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\, x_1^2 + B\, x_1 x_2 + C\, x_2 x_1 + D\, x_2^2.$$

5.1.2 Non-commutative coefficients

The generalized quadratic polynomial with non-commutative coefficients is a trifle more complicated. However, in this paper, we will be concerned mostly with the case $g = 1$. That is, we will need to extract only one letter, h, from the “middle” matrix P.

The general polynomial quadratic in h, with non-commutative coefficients is written

$$Q(h) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, (h\, m_i)^T\, c_{ij}\, (h\, m_j)$$

here m_i are the terms applied “outside” the target letters h, and c_{ij}, are the “inside” terms and A_{ij} are scalars. These terms do not necessarily commute, so they can not be grouped together and simplified as above. Define A as the N-by-N matrix whose i,j^{th} element is $A_{ij}\, c_{ij}$, and define q as

$$q = h(m_1,\, m_2,\, \dots,\, m_N).$$

Then we see that

$$q^T P q = \sum_{i=1}^{n}\sum_{j=1}^{n}(q)_i^T P_{ij} (q)_j = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij},\, (h\, m_i)^T\, c_{ij}\, (h\, m_j).$$

Example 5.2 *A matrix representation with non-commutative coefficients*

$$3\, x_1 h x_2^2 h x_1 + h x_1 x_2 x_1 h - h^2 x_2^2 + h x_2^3 x_1 h x_2^2 + 5\, x_1 x_2 h x_2 h x_2 x_1$$
$$= \begin{pmatrix} h \\ hx_1 \\ hx_2x_1 \\ hx_2^2 \end{pmatrix}^T \begin{pmatrix} x_1x_2x_1 & 0 & 0 & -1 \\ 0 & 3\,x_2^2 & 0 & 0 \\ 0 & 0 & 5\,x_2 & 0 \\ x_2^3 x_1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ hx_1 \\ hx_2x_1 \\ hx_2^2 \end{pmatrix} = q^T P q.$$

It can be easily verified that Q is symmetric if and only if the matrix P is symmetric.

5.1.3 Convexity of p vs. Positivity of its Middle Matrix

Given the polynomial $Q(x,h)$ which is quadratic in h there may be several $q^T P q$ representations of Q. However, once q is chosen, with entries which are monomials that do not repeat, the middle matrix P is uniquely determined (up to adding on rows and columns of zeroes). Furthermore, the polynomial Q in x and h is matrix positive if and only if $P(x)$ is a positive semidefinite matrix whenever symmetric matrices are substituted for x. Thus checking matrix positivity of Q is equivalent to checking matrix positivity of P. For discussion and proofs of all this, see [CHSY03].

5.2 The Zeroes Lemma

The following is useful in our analysis of subharmonics.

Lemma 5.1 *Let $A \in R^{N\times N}$ be any symmetric matrix with entries in some domain R of polynomials on free variables with real coefficents. If there exists some diagonal entry $a_{ii} = 0$ and corresponding off-diagonal entries $a_{ij} = a_{ji}^T \neq 0$, then A is not matrix-positive semidefinite*

Proof:
Let e_i and e_j be standard basis vectors for R^N (i.e. $e_i^T A\, e_j = a_{ij}$) and define $v := \beta_1 e_i + \beta_2 e_j$ where $\beta_1,\, \beta_2 \in \mathbb{R}$. Then,

$$v^T A v = ((a_{ij} \, + \, a_{ji})\, \beta_1 + a_{jj}\, \beta_2)\, \beta_2 = (2\, a_{ij}\, \beta_1 + a_{jj}\, \beta_2)\, \beta_2$$

Given $\beta_2 > 0$, we can choose β_1 such that

$$2\, \beta_1\, a_{ij} \beta_2 + \beta_2 a_{jj}\, \beta_2$$

is either matrix-positive or -negative.

■

This lemma is very useful when applied to our matrix representation of the Laplacian of a symmetric polynomial on free variables. Note that we can express the Laplacian, $\mathrm{Lap}(p,h)$, of any polynomial $p(x)$ as:

$$\mathrm{Lap}[p,h] = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, (h\, m_i)^T\, c_{ij}\, (h\, m_j)$$

and that $\mathrm{Lap}[p,h] = q^T P q$, where $p_{ij} = A_{ij} c_{ij}$ and

$$q = (h\, m_1,\, \ldots,\, h\, m_n)^T .$$

For any i, if

$$A_{ii}\, (hm_i)^T\, c_{ii}\, (hm_i) = 0,$$

then

$$p_{ii} = A_{ii}\, c_{ii} = 0.$$

By the Zeroes Lemma, either P is not positive-semidefinite ($\mathrm{Lap}[p,h]$ is not matrix-positive) or

$$p_{ij} = p_{ji} = 0 \quad \text{for} \quad 1 \leq j \leq n$$

in the latter case,

$$A_{ij}\, (hm_i)^T\, c_{ij}\, (hm_j) = A_{ji}\, (hm_j)^T\, c_{ji}\, (hm_i) = 0 \quad \text{for} \quad 1 \leq j \leq n.$$

5.3 The Spanning Polynomials for Subharmonics

We begin with a basis for the set of degree 4 homogeneous polynomials in symmetric free variables

$$\begin{aligned} p = A_1\, x_1^4 x_1^2 + A_2(x_1^3 x_2 + x_2 x_1^3) \\ + A_3\,(x_1^2 x_2 x_1 + x_1 x_2 x_1^2) + A_4\,(x_1^2 x_2^2 + x_2^2 x_1^2) \\ + A_5\,(x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1) + A_6\, x_1 x_2^2 x_1 \\ + A_7\,(x_1 x_2^3 + x_2^3 x_1) + A_8\, x_2 x_1^2 x_2 \\ + A_9\,(x_2 x_1 x_2^2 + x_2^2 x_1 x_2) + A_{10}\, x_2^4. \end{aligned}$$

We calculate the Laplacian of p:

$$\begin{aligned} &2\,A_1\,(h^2x_1^2 + hx_1hx_1 + hx_1^2h + x_1h^2x_1 + x_1hx_1h + x_1^2h^2) \\ &+ 2\,A_2\,(h^2x_1x_2 + hx_1hx_2 + x_1h^2x_2 + x_2h^2x_1 + x_2hx_1h + x_2x_1h^2) \\ &+ 2\,A_3\,(h^2x_2x_1 + hx_1x_2h + hx_2hx_1 + hx_2x_1h + x_1hx_2h + x_1x_2h^2) \\ &+ 2\,A_4\,(h^2x_1^2 + h^2x_2^2 + x_1^2h^2 + x_2^2h^2) \\ &+ 2\,A_5\,(hx_1hx_1 + hx_2hx_2 + x_1hx_1h + x_2hx_2h) + \ 2\,A_6\,(hx_2^2h + x_1h^2x_1) \\ &+ 2\,A_7\,(h^2x_2x_1 + hx_2hx_1 + x_1h^2x_2 + x_1hx_2h + x_1x_2h^2 + x_2h^2x_1) + \ 2\,A_8\,(hx_1^2h + x_2h^2x_2) \\ &+ 2\,A_9\,(h^2x_1x_2 + hx_1hx_2 + hx_1x_2h + hx_2x_1h + x_2hx_1h + x_2x_1h^2) \\ &+ 2\,A_{10}\,(h^2x_2^2 + hx_2hx_2 + hx_2^2h + x_2h^2x_2 + x_2hx_2h + x_2^2h^2). \end{aligned}$$

5.3.1 The Laplacian Matrix Decomposition

The directional Laplacian is quadratic in h, and so can be represented by the matrix product $q^T Pq$, where q and $P/2$ are

$$\begin{pmatrix} h & x_1\,h & x_2\,h & x_1^2\,h & x_1\,x_2\,h & x_2\,x_1\,h & x_2^2 h \end{pmatrix}^T \quad \text{and}$$

$$\begin{pmatrix} (A_1+A_8)\,x_1^2+(A_6+A_{10})\,x_2^2 \\ +(A_3+A_9)\,(x_1x_2+x_2x_1) & (A_1+A_5)\,x_1 + (A_3+A_7)\,x_2 & (A_2+A_9)\,x_1 + (A_5+A_{10})\,x_2 & A_1{+}A_4 & A_3{+}A_7 & A_2{+}A_9 & A_4{+}A_{10} \\ (A_1+A_5)\,x_1 + (A_3+A_7)\,x_2 & A_1+A_6 & A_2+A_7 & 0 & 0 & 0 & 0 \\ (A_2+A_9)\,x_1 + (A_5+A_{10})\,x_2 & A_2+A_7 & A_8+A_{10} & 0 & 0 & 0 & 0 \\ A_1+A_4 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ A_3+A_7 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ A_2+A_9 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ A_4+A_{10} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{pmatrix}$$

respectively. We wish to find the conditions on $\{A_1, \ldots, A_{10}\}$ that ensure that this matrix is positive-semidefinite. By the Zeroes Lemma, the zeroes on the last four diagonals force all entries in the last four rows or columns to be zero. This corresponds to the following assignments:

$$A_4 = -A_1, \quad A_{10} = A_1, \quad A_9 = -A_2, \quad A_7 = -A_3. \tag{44}$$

Applying these conditions to the matrix above, and ignoring the zeroed-out rows and columns, we have:

$$\begin{pmatrix} (A_1+A_8)\,x_1^2+(A_1+A_6)\,x_2^2 + (A_2-A_3)(x_2\,x_1-x_1\,x_2) & (A_1{+}A_5)x_1 & (A_1{+}A_5)x_2 \\ (A_1{+}A_5)x_1 & A_1{+}A_6 & A_2{-}A_3 \\ (A_1{+}A_5)x_2 & A_2{-}A_3 & A_1{+}A_8 \end{pmatrix}.$$

This matrix can be simplified by substitution of recurring pairs by single letters:

$$G = A_1 + A_6, \quad H = A_1 + A_8 \quad J = A_2 - A_3, \quad K = A_1 + A_5.$$

Now we have a matrix small enough that we can find its LDL^T (Cholesky) decomposition

$$\begin{pmatrix} Hx_1^2 - J(x_1x_2 + x_2x_1) + Gx_2^2 & Kx_1 & Kx_2 \\ Kx_1 & G & J \\ Kx_2 & J & H \end{pmatrix}.$$

5.3.2 Conditions for Positivity

It is now possible to discover the exact conditions for matrix positivity. Each diagonal entry of D must be equal to or greater than zero for the Laplacian to be matrix-positive). This will reveal the values of G, H, J and K such that the above is positive-semidefinite.

$$\begin{pmatrix} G & 0 & 0 \\ 0 & H-\frac{J^2}{G} & 0 \\ 0 & 0 & Hx_1^2+J(x_1x_2+x_2x_1)+Gx_2^2-\frac{K^2x_1^2}{G}-\frac{(\frac{JKx_1}{G}+Kx_2)(\frac{JKx_1}{G}+Kx_2)}{H-\frac{J^2}{G}} \end{pmatrix}.$$

And so we are left with three inequalities, which must be satisfied for matrix positivity:

$$G>0, \quad H-\frac{J^2}{G}>0,$$

$$Hx_1^2+J(x_1x_2+x_2x_1)+Gx_2^2-\frac{K^2x_1^2}{G}-\frac{(\frac{JKx_1}{G}+Kx_2)(\frac{JKx_1}{G}+Kx_2)}{H-\frac{J^2}{G}}>0.$$

The last condition involves x_1 and x_2, and therefore must itself be converted into a matrix to reveal the conditions on G, H, J and K under which it is positive. [CHSY03]

$$\begin{pmatrix} G-\frac{H^2K^2}{(G-\frac{H^2}{J})J^2}-\frac{K^2}{J} & H+\frac{HK^2}{(G-\frac{H^2}{J})J} \\ H+\frac{HK^2}{(G-\frac{H^2}{J})J} & J-\frac{K^2}{G-\frac{H^2}{J}} \end{pmatrix}.$$

Again we perform the LDL^T decomposition:

$$\begin{pmatrix} G-\frac{K^2}{H-\frac{J^2}{JG}} & 0 \\ 0 & H-\frac{J^2K^2}{(H-\frac{J^2}{G})G^2}-\frac{K^2}{G}-\frac{\left(J-\frac{JK^2}{(H-\frac{J^2}{G})G}\right)^2}{G-\frac{K^2}{H-\frac{J^2}{G}}} \end{pmatrix}.$$

Although the inequality

$$H-\frac{J^2K^2}{(H-\frac{J^2}{G})G^2}-\frac{K^2}{G}-\frac{\left(J-\frac{JK^2}{(H-\frac{J^2}{G})G}\right)^2}{G-\frac{K^2}{H-\frac{J^2}{G}}}>0 \tag{45}$$

is quite complicated, we can simplify it some by multiplying it by expressions which are known to be positive, such as:

$$G, \quad H - \frac{J^2}{G}, \quad \text{and} \quad G - \frac{K^2}{H - \frac{J^2}{G}}$$

which we encountered earlier. This gives a polynomial inequality equivalent to (45), which, after some simplification, gives us:

$$(GH - J^2 - K^2)^2 > 0.$$

Which, considering only the case of all real coefficients, is rather vacuous, informing us only that $GH - J^2 - K^2 \neq 0$.

Bringing all our inequalities together (simplifying each as we did above), we notice that two are redundant :

$$(I)\; G > 0, \qquad (II)\; GH > J^2, \qquad (III)\; GH > J^2+K^2, \qquad (IV)\; GH \neq H^2+K^2$$

as (III) implies (II) and (IV).

Therefore, we conclude that the set of polynomials making the Laplacian matrix "positive" is exactly those of the form:

$$\begin{aligned} f =& A_1(x_1^4 - x_1^2x_2^2 - x_2^2x_1^2 + x_2^4) \\ &+ A_2(x_1^3x_2 + x_2x_1^3 - x_2x_1x_2^2 - x_2^2x_1x_2) \\ &+ A_3(x_1^2x_2x_1 + x_1x_2x_1^2 - x_1x_2^3 - x_2^3x_1) \\ &+ A_5(x_1x_2x_1x_2 + x_2x_1x_2x_1) \\ &+ A_6\, x_1x_2^2x_1 + A_8\, x_2x_1^2x_2 \end{aligned} \tag{46}$$

with coefficients satisfying the inequalities:

$$(A_1 + A_8)(A_1 + A_6) > (A_3 - A_2)^2 + (A_1 + A_5)^2 \tag{47}$$

$$\text{and} \quad A_1 + A_8 > 0 \quad (\text{or, equivalently} \quad A_1 + A_6 > 0). \tag{48}$$

6 Future Work

The subharmonic polynomials may serve as substitutes for convex polynomials in special cases. Currently, our group is looking at other weakenings of convexity, including "quasiconvexity," in which the number of negative eigenvalues are tolerably small.

The proofs in progress that $\mathrm{Re}(\gamma^d)$ and $\mathrm{Im}(\gamma^d)$ (resp. $[\mathrm{Re}(\gamma^d)]^2$ and $[\mathrm{Im}(\gamma^d)]^2$) are the only harmonic (resp. subharmonic) polynomials of degree d (resp. $2d$ when $d > 2$) relies on an observation on the words generated by substituting pairs of identical letters (as with the Laplacian). These proofs should follow shortly.

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