

Variational methods for weak solutions to the Einstein  
Hamiltonian constraint on finite domains with boundary

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## Abstract

Einstein's theory of general relativity is described mathematically by a particular coupled system of ten time-dependent non-linear partial differential equations (PDEs). These equations describe how the presence of matter and energy curve space-time and give rise to what we experience as gravity. Although mathematically elegant, these equations are extremely difficult to solve (either analytically or computationally) due to the complex coupled non-linear nature of the PDEs.

We begin by giving derivations of the system via Lagrangian formalism. From there we discuss the 3 + 1 splitting of the system which gives rise to the six dynamical equations and the four spatial "constraints." We will then focus our attention for the remainder of the paper on one of these four constraints, collectively referred to as the *Hamiltonian* energy constraint. Working on a compact Riemannian 3-manifold  $(\Sigma^3, \bar{g}_{ij})$  with boundary, we derive a priori  $L^\infty$ -bounds on solutions to the Hamiltonian constraint, and establish the existence of weak solutions.

This thesis is part of a larger joint work with M. Holst and G. Nagy.

# Acknowledgement

Marcus Tullius Cicero had it right when he remarked,

*futura sciri non possunt.*

We do not know why things happen the way they do, but I feel forever indebted to the events which, and to whomever, led me to where I am today. I would like to especially thank Mike, for inspiring me so; for without him, I may have never made the switch to study mathematics. And recently, to G. Nagy and R. Szypowski for being so patient and generous through discussions which made this entire project possible. Finally, two persons who helped pushed me to better my mathematics in the classroom: L.W. Small and J. Farina.

למשפחה וחברים שלי

יונתן דוד קוממי

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# Chapter 1

## Notation and Conventions

We will denote as  $\Sigma^3$  a space-like hypersurface of a Lorentzian 4-manifold  $(\mathcal{M}, g_{\mu\nu})$ . Given the imbedding  $\iota : \Sigma^3 \hookrightarrow \mathcal{M}$ , the induced Riemannian metric  $\bar{g}$  is given by  $\iota^*g$ . Let  $\bar{\nabla}$  be the unique torsion-free affine connection of the metric  $\bar{g}$ . In local co-ordinates,

$$\bar{\nabla}_j \xi^i = \partial_j \xi^i + \Gamma^i_{jk} \xi^k. \quad (1.1)$$

We follow the convention where Greek letters are summed over 0, 1, 2, 3 and Roman letters over 1, 2, 3.

### Domain and Boundary

Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded set with non-empty boundary  $\partial\Omega$  satisfying the strong local Lipschitz property [1]. We will assume the boundary  $\partial\Omega$  can be divided into a Neumann part,  $\partial\Omega_N$ , and a Dirichlet part,  $\partial\Omega_D \neq \emptyset$ , such that

$$\begin{aligned} \partial\Omega &= \partial\Omega_D \cup \partial\Omega_N \\ \emptyset &= \bar{\partial\Omega}_D \cap \bar{\partial\Omega}_N. \end{aligned}$$

So note in particular that we allow for the case that the Neumann boundary has measure zero and we have Dirichlet conditions on the entire boundary.

### $L^p$ , Sobolev spaces and their Embeddings

Let  $\Omega$  be a measurable set. Then the set of  $p$ -integrable functions is defined as

$$L^p(\Omega, \mathbb{R}^n) = \left\{ f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R} : \|f\|_{0,p} := \left( \int_{\Omega} |f|^p d\mu \right)^{1/p} < \infty \right\}$$

for  $1 \leq p < +\infty$ . When  $p = \infty$ , we define

$$L^\infty(\Omega, \mathbb{R}^n) = \{ f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R} : \|f\|_{0,\infty} := \text{ess sup } |f| \}$$

where  $\text{ess sup } |f| = \inf \{ a \in \mathbb{R} : \mu(\{|f(x)| > a\}) = 0 \}$ .

By defining the equivalence relation  $f \sim g \iff \|f - g\|_{0,p} = 0$ ,  $L^p(\Omega, \mathbb{R}^n) / \sim$  becomes a Banach space. Moreover, for  $p = 2$ , the inner-product

$$(f, g) := \int_{\Omega} fg \, d\mu = \|fg\|_{0,1} \quad (1.2)$$

makes  $L^2(\Omega)$  into a Hilbert space.

Now let  $W^{k,p}(\Omega, \mathbb{R}^n)$  be the Sobolev spaces of functions in  $L^p(\Omega, \mathbb{R}^n)$  with  $k$  covariant derivatives in  $L^p(\Omega, \mathbb{R}^n)$ . That is,

$$W^{k,p}(\Omega, \mathbb{R}^n) = \{f \in L^p(\Omega, \mathbb{R}^n) : \partial^\alpha f \in L^p(\Omega, \mathbb{R}^n) \text{ for all multi-indices } |\alpha| \leq k\}. \quad (1.3)$$

Endowed with the norm

$$\|f\|_{k,p} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{0,p} \quad (1.4)$$

$W^{k,p}(\Omega, \mathbb{R}^n)$  becomes a Banach space.

Introducing the continuous extensions of scalar fields to the boundary, we define

$$\begin{aligned} \text{tr}_D : W^{k,p}(\Omega, \mathbb{R}^n) &\rightarrow W^{k-1/p,p}(\partial\Omega_D, \mathbb{R}^n) \\ \phi &\mapsto \phi|_{\partial\Omega_D} \end{aligned} \quad (1.5)$$

for  $k > 1/p$ . Similarly, we define  $\text{tr}_N$ .

For a Sobolev space  $W^{k,p}(\Omega, \mathbb{R}^n)$ , let  $p^*$  denote its conjugate exponent, i.e. let  $p^* = \frac{np}{n-pk}$ . If  $\frac{1}{p} > \frac{k}{n}$  then we have the continuous embedding

$$W^{k,p}(\Omega, \mathbb{R}^n) \hookrightarrow L^q(\Omega, \mathbb{R}^n), \quad q = p^*.$$

Moreover, the embedding is compact for  $q < p^*$ . Note for  $k = 0$ , we have  $W^{0,p}(\Omega, \mathbb{R}^n) := L^p(\Omega, \mathbb{R}^n)$  and

$$L^q(\Omega, \mathbb{R}^n) \hookrightarrow L^p(\Omega, \mathbb{R}^n)$$

for all  $p < q$ . A good resource for these, and similar, results on Sobolev spaces can be found in [1].

## Chapter 2

# Introduction

One of the most interesting problems in general relativity today is to find approximations of solutions to the Einstein equations and to compare the results of such calculations to the expected data from interferometric gravitational wave observatories, such as the NSF-funded LIGO project. Such observatories are expected to make observations of “burst” events, mainly supernovae and collisions of compact objects, e.g. black holes and neutron stars. Modeling such phenomena in 3+1 space-time dimensions, whether it be analytically or numerically, is extremely difficult. Even after the effects of matter are eliminated, a solution to the equations is still quite challenging to come by. Certain problems in 1+1 and 2+1 dimensions (spherical- and axisymmetry) have seen much success numerically, but no such analogue exists in the full 3+1 problem. A review article of numerical approaches can be found in [16].

As is well known, solutions to the Einstein equations are constrained in a manner similar to Maxwell’s equations in that the initial data for a particular space-time must satisfy a set of spatial equations which are then preserved throughout the evolution. Just as in the electromagnetism case, these constraints may be put in a purely elliptic form. The **York conformal decomposition** produces a covariant non-linear elliptic system on a 3-manifold, consisting of a non-linear scalar Hamiltonian constraint coupled to a linear vector momentum constraint.

We shall see in more detail in §4.1 that the coupled Hamiltonian and momentum constraints can be written in the form:

$$-\hat{\Delta}\phi + \frac{\hat{R}}{8}\phi + \frac{1}{12}\tau\phi^5 - \frac{1}{8}[\hat{\sigma}_* + (\hat{\mathcal{L}}\mathbf{w})]^2\phi^{-7} - 2\pi\hat{\rho}\phi^{-3} = 0 \quad (2.1)$$

$$\hat{\nabla}_j(\hat{\mathcal{L}}\mathbf{w})^{ij} = \frac{2}{3}\phi^6\hat{g}^{ij}\hat{\nabla}_j\tau + 8\pi\hat{j}^j \quad (2.2)$$

with the addition of the appropriate boundary conditions in the case of domains with boundary, for the unknown pair of functions  $(\phi, \mathbf{w})$ . Usually, the manifold is unbounded (or, compact without boundary); however, our interest is in an explicit boundary-value formulation on a compact 3-manifold with boundary, which is a required formulation before standard numerical approximation methods for elliptic systems can be applied.



## Chapter 3

# Einstein Field Equations

### 3.1 History

In 1915, Albert Einstein published his groundbreaking work *Die Grundlage allgemeinen Relativitätstheorie* (The Foundation of the General Theory of Relativity). Because this debuted at a time when many were working on unifying quantum mechanics and electromagnetism, not much thought was given to Einstein's theory except for the fact that it corrected Newtonian laws of gravitation to be compatible with special relativity. However, once astronomical discoveries were made starting in the 1960s, the full implications of general relativity (GR) were finally being realized. In particular, were those discoveries of stellar collapse and explosion in the Universe.

Einstein viewed the fabric of space-time as a Lorentzian 4-manifold  $(\mathcal{M}, g_{\mu\nu})$  which warps under the presence of matter and energy, Fig. 3.1. An important implication of GR, not present in the Newtonian laws, is that gravity is to be described as the manifestation of the warping of  $\mathcal{M}$  under the presence of matter. Thus the mantra:

*“Matter tells space-time how to curve and curved space-time tells matter how to move”*

In the next section we will formalize this statement by deriving the Einstein Field Equations from a Lagrangian, using the Principle of Least Action.

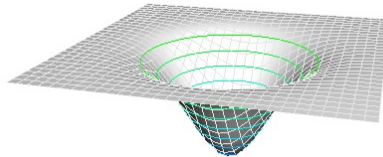


Figure 3.1: Curved space-time

## 3.2 Lagrangian Formalism

### Canonical Measure

Let  $(\mathcal{M}, g_{\mu\nu})$  be a pseudo-Riemannian 4-manifold, and let  $\eta := \sqrt{-|g|} dx^0 \wedge \dots \wedge dx^3$  be the canonical volume element, where  $|g| := \det g_{\mu\nu}$ . We can define a linear functional  $\phi : C_0^0(\mathcal{M}) \rightarrow \mathbb{R}$  by  $f \mapsto \int_{\Omega} f \eta$ , where  $C_0^0(\mathcal{M})$  is the set of  $C^0(\mathcal{M})$  functions with compact support. By the Reisz Representation Theorem a positive (Borel) measure  $dv_g$  exists on the  $\sigma$ -algebra of Borel sets on  $\mathcal{M}$ ; we call this measure the **canonical** or **Riemannian measure** on  $\mathcal{M}$ . In local coordinates,

$$dv_g = \sqrt{-|g|} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 =: \sqrt{-|g|} d^4x.$$

### 3.2.1 Einstein-Hilbert Action (Vacuum Equations)

To derive the vacuum Einstein equations we consider the **Lagrangian density**

$$\mathfrak{L}_G(g) = R(g) \sqrt{-|g|} \tag{3.1}$$

where  $R(g) := g^{\mu\nu} R_{\mu\nu}$  is the Ricci scalar of the metric  $g$ . Now define the **Einstein-Hilbert action**

$$\begin{aligned} S : \{C^2 \text{ metrics on } \mathcal{M}\} &\rightarrow \mathbb{R} \\ g^{\mu\nu} &\mapsto \int_{\Omega} \mathfrak{L}_G(g) d^4x. \end{aligned} \tag{3.2}$$

According to the Principle of Least Action for a field theory, the idea then is that the Einstein equations correspond to the stationary points of  $S(g)$ . To show this, let us compute the first variation  $\delta S(g)$ . The following discussion was adapted from [17, 18]. Using the calculus of variations, we introduce a one-parameter family of (inverse) metrics  $g_t$  and compute  $\delta S(g) = \frac{d}{dt} S(g^{\mu\nu} + tg_t)|_{t=0}$ . Doing so, we have

$$\begin{aligned} \delta S(g) &= \delta \int_{\Omega} R(g) \sqrt{-|g|} d^4x \\ &= \int_{\Omega} \delta(g^{\mu\nu} R_{\mu\nu}(g) \sqrt{-|g|}) d^4x. \end{aligned} \tag{3.3}$$

Using the product rule (twice) and then re-contracting the Ricci tensor, (3.3) becomes

$$\int_{\Omega} \delta R_{\mu\nu}(g) g^{\mu\nu} \sqrt{-|g|} d^4x + \int_{\Omega} R(g) \delta \sqrt{-|g|} d^4x + \int_{\Omega} R_{\mu\nu}(g) \sqrt{-|g|} \delta g^{\mu\nu} d^4x. \tag{3.4}$$

In order to simplify (3.4), we establish some results on the variations of the Ricci tensor, the metric tensor and its determinant.

#### Variation of the Ricci Tensor $\delta R_{\mu\nu}$

Begin by recalling the definition of the **Ricci tensor**  $R_{\mu\nu}$ :

$$R_{\mu\nu} = \partial_{\lambda} \Gamma^{\lambda}_{\mu\nu} - \partial_{\nu} \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\nu} \Gamma^{\lambda}_{\sigma\lambda} - \Gamma^{\sigma}_{\mu\lambda} \Gamma^{\lambda}_{\sigma\nu}, \tag{3.5}$$

with the **affine Christoffel connections**:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} (\partial_{\sigma} g_{\nu\lambda} + \partial_{\sigma} g_{\lambda\nu} - \partial_{\nu} g_{\lambda\sigma}). \tag{3.6}$$

By defining the usual exponential map  $\exp_{(p)} : \mathcal{T}_p\mathcal{M} \rightarrow \mathcal{M}$  by  $v \mapsto \gamma_v(1)$  for geodesics  $\gamma_v$  and using the Implicit Function Theorem we have a very useful result from differential geometry that states

**Proposition 3.2.1.** *The mapping  $\exp_{(p)}$  is a diffeomorphism from a neighborhood of  $0 \in \mathcal{T}_p(\mathcal{M})$  to a neighborhood of  $p \in \mathcal{M}$ .*

In particular, this allows us to choose a set of (normal) coordinates for which the affine connections  $\Gamma^\rho_{\mu\nu}$  vanish at the point  $p$ . This follows from the fact that for geodesics,

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0. \quad (3.7)$$

Thus when we take the first variation of  $R_{\mu\nu}$ , all terms which do not have a partial derivative vanish. That is, we may write the first variation of the Ricci tensor as

$$\delta R_{\mu\nu}(g) = \partial_\lambda[\delta\Gamma^\lambda_{\mu\nu}] - \partial_\nu[\delta\Gamma^\lambda_{\mu\lambda}]. \quad (3.8)$$

Now note that the affine connections  $\Gamma^\mu_{\nu\lambda}$  become tensors once we take their variations. As a result, we may replace  $\partial$  by the covariant derivative  $\nabla$ . We are left with what is known as the **Palatini identity**:

$$\delta R_{\mu\nu}(g) = \nabla_\lambda[\delta\Gamma^\lambda_{\mu\nu}] - \nabla_\nu[\delta\Gamma^\lambda_{\mu\lambda}]. \quad (3.9)$$

### Variation of the Metric Tensor and its Determinant $\delta g^{\mu\nu}$ , $\delta\sqrt{-|g|}$

First note that we have  $g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$ , since  $0 = \delta 4 = \delta(g^{\mu\nu}g_{\mu\nu}) = g_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu}$ . Now compute (using the product rule) to find

$$\delta\sqrt{-|g|} = -\frac{1}{2} \frac{1}{\sqrt{-|g|}} \delta|g|. \quad (3.10)$$

Finally, to compute the variation of the determinant, we use Cramer's rule to note that  $|g|g_{\mu\nu}$  is the cofactor of  $g^{\mu\nu}$ . And so,  $\delta|g| = |g|g_{\mu\nu}\delta g^{\mu\nu}$ . Using this we can rewrite (3.10) as

$$\delta\sqrt{-|g|} = -\frac{1}{2} \sqrt{-|g|} g_{\mu\nu} \delta g^{\mu\nu}. \quad (3.11)$$

Combining (3.9) and (3.11) together, we may write (3.4) as

$$\begin{aligned} \delta S(g) &= \int_{\Omega} \delta R_{\mu\nu}(g) g^{\mu\nu} \sqrt{-|g|} d^4x + \int_{\Omega} R(g) \delta\sqrt{-|g|} d^4x + \int_{\Omega} R_{\mu\nu}(g) \sqrt{-|g|} \delta g^{\mu\nu} d^4x \\ &= \int_{\Omega} (\nabla_\lambda[\delta\Gamma^\lambda_{\mu\nu}] - \nabla_\nu[\delta\Gamma^\lambda_{\mu\lambda}]) g^{\mu\nu} \sqrt{-|g|} d^4x \\ &\quad + \int_{\Omega} R(g) \left(-\frac{1}{2} \sqrt{-|g|} g_{\mu\nu} \delta g^{\mu\nu}\right) d^4x + \int_{\Omega} R_{\mu\nu}(g) \sqrt{-|g|} \delta g^{\mu\nu} d^4x \\ &= \int_{\Omega} \nabla^\mu V^\mu \sqrt{-|g|} d^4x + \int_{\Omega} \left(R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu}\right) \sqrt{-|g|} \delta g^{\mu\nu} d^4x, \end{aligned} \quad (3.12)$$

where  $V^\mu := g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} - g^{\lambda\nu} \delta\Gamma^\mu_{\mu\nu}$ . According to Stokes's Theorem, the first integral on the right contributes only a boundary term. Therefore, on closed manifolds and for variations  $\delta g^{\mu\nu}$  with compact support in space-time we have

$$\delta S(g) = \int_{\Omega} \sqrt{-|g|} \left(R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu}\right) \delta g^{\mu\nu} d^4x. \quad (3.13)$$

As a stationary point, we have  $(R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu}) = 0$ , the Einstein Field Equations in vacuum. For manifolds which are not closed, or where the variations do not have compact support, it is possible to re-define the Hilbert action to negate the contribution of the boundary integral.

Similarly, one can derive the Equations with energy source terms from a Lagrangian by adding an additional term  $\mathfrak{L}_M$  to  $\mathfrak{L}_G$ . This term can be used to define the **stress-energy tensor**  $T_{\mu\nu}$ , which encodes information about the flux and density of energy and momentum in local space-time [18]. In this way we derive the general form of the Einstein Field Equations:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \quad (3.14)$$

### 3.3 Dynamical 3+1 Formulation

Although deriving the Einstein Equations from a Lagrangian is quite elegant as we see above, its practicality is somewhat limited. In this section, we will introduce what is known as the 3+1 splitting of the equations governing dynamical fields on space-like hypersurfaces which evolve over time. This formalism allows one to identify the six dynamic equations and four constraints. These equations themselves are at the cornerstone of present-day theoretical, and numerical, investigations.

One drawback, however, is that the splitting is deeply rooted in geometry and much machinery will have to be developed in order to give a genuine discussion of the details. Nevertheless, it will prove to be insightful and useful for future study.

#### 3.3.1 Cartan Structure Equations

Let  $\{e_\mu\}$  be an orthonormal basis of  $\mathcal{TM}$ . To each basis element, we associate a corresponding orthogonal dual element  $\vartheta^\mu$  such that

$$\vartheta^\nu(e_\mu) = \delta^\nu_\mu. \quad (3.15)$$

We define the **connection forms**  $\omega^\mu_\nu \in \Lambda^1$  (totally anti-symmetric 1-forms) such that

$$\nabla_X e_\nu = \omega^\mu_\nu(X)e_\mu. \quad (3.16)$$

From here it follows that  $\nabla_X \vartheta^\mu = -\omega^\mu_\nu(X)\vartheta^\nu$ , since  $0 = \nabla_X \vartheta^\nu(e_\mu)$ . Using the Leibniz rule

$$\begin{aligned} 0 = \nabla_X \vartheta^\nu(e_\mu) &= \nabla_X \vartheta^\mu(e_\nu) + \vartheta^\mu(\nabla_X e_\nu) \\ &= \nabla_X \vartheta^\mu(e_\nu) + \vartheta^\mu(\omega^\rho_\nu(X)e_\rho) \\ &= \nabla_X \vartheta^\mu(e_\nu) + \omega^\mu_\nu(X)e_\mu. \end{aligned} \quad (3.17)$$

There are two important vector fields on  $\mathcal{M}$  which induce connection forms vital to the 3+1 splitting. Denote the class of all smooth vectors fields on  $\mathcal{M}$  by  $\mathfrak{X}(\mathcal{M})$ .

**Definition 3.3.1.** Let  $\nabla$  be an affine connection on  $\mathcal{M}$ . We define torsion as

$$\begin{aligned} T : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\rightarrow \mathfrak{X}(\mathcal{M}) \\ X \times Y &\mapsto \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned} \quad (3.18)$$

and curvature as

$$\begin{aligned}
R &: \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(\mathcal{M}) \\
X \times Y \times Z &\mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\end{aligned} \tag{3.19}$$

where  $[\cdot, \cdot]$  is the commutator of two vector fields.

In fact, connections derived from metrics are generally constructed to have vanishing torsion. There are, however, theories of gravitation in which this is not true [18]. Define the torsion form as the connection  $\Theta^\mu \in \bigwedge^2$  (totally anti-symmetric 2-forms) such that

$$0 = T(X, Y) = \Theta^\mu(X, Y)e_\mu.$$

Similarly we define the curvature form as the connection  $\Omega^\mu{}_\nu \in \bigwedge^2$  such that

$$R(X, Y)e_\nu = \Omega^\mu{}_\nu(X, Y)e_\mu.$$

Using the definitions of torsion and curvature, we can work out what the connection forms are. In local co-ordinates, we have the torsion connection

$$\begin{aligned}
0 = \Theta^\mu(X, Y)e_\mu &= \nabla_X Y - \nabla_Y X - [X, Y] \\
&= \nabla_X(\vartheta^\nu(Y)e_\nu) - \nabla_Y(\vartheta^\nu(X)e_\nu) - \vartheta^\nu([X, Y])e_\nu \\
&= X\vartheta^\nu(Y)e_\nu + \omega^\mu{}_\nu(X)e_\mu - Y\vartheta^\nu(X)e_\nu - \omega^\mu{}_\nu(Y)e_\mu - \vartheta^\nu([X, Y])e_\nu \\
&= [X\vartheta^\nu(Y) - Y\vartheta^\nu(X) - \vartheta^\nu([X, Y])]e_\nu + [\omega^\mu{}_\nu(X) - \omega^\mu{}_\nu(Y)]e_\mu \\
&= d\vartheta^\mu(X, Y)e_\mu + (\omega^\mu{}_\nu \wedge \vartheta^\nu)(X, Y)e_\mu
\end{aligned} \tag{3.20}$$

and similarly we have the curvature form

$$\Omega^\mu{}_\nu(X, Y)e_\mu = d\omega^\mu{}_\nu(X, Y)e_\mu + (\omega^\mu{}_\gamma \wedge \omega^\gamma{}_\nu)(X, Y)e_\mu \tag{3.21}$$

where  $d$  is the exterior derivative (§7 Appendix). Equations (3.20) and (3.21) are known as the **first** and **second Cartan structure equations**, respectively [17, 18].

As a note, in the case where we choose  $e_\mu := \partial/\partial x^\mu$  and  $\vartheta^\mu := dx^\mu$ , the components of  $\Omega^\mu{}_\nu$  define the Riemann tensor,  $R^\mu{}_{\nu\sigma\rho}$ :

$$\begin{aligned}
\langle dx^\mu, R(\partial_\sigma, \partial_\rho)\partial_\nu \rangle &= \langle dx^\mu, \Omega^\delta{}_\nu(\partial_\sigma, \partial_\rho)\partial_\delta \rangle \\
&= \Omega^\mu{}_\nu(\partial_\sigma, \partial_\rho) \\
&=: R^\mu{}_{\nu\sigma\rho}
\end{aligned} \tag{3.22}$$

so that we have  $\Omega^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\sigma\rho}dx^\sigma \wedge dx^\rho$ .

Consider now these equations restricted to Riemannian 3-submanifolds  $\Sigma^3 \hookrightarrow \mathcal{M}$ . Particularly important to our derivation of the 3 + 1 splitting will be to figure exactly how the torsion and curvature forms are related on  $\mathcal{M}$  to  $\Sigma^3$ . Beginning with the first structure equation, we will derive the equations of Gauss and Weingarten which relate the torsion forms. Similarly, we end with the equations of Gauss and Codazzi-Mainardi which relate the curvature forms. Recall that the induced metric on  $\Sigma^3$  is defined as  $\bar{g} := \iota^*g$ , for the given embedding into  $\mathcal{M}$ . As a way to distinguish quantities, e.g. metric, basis on  $\Sigma^3$  from that of  $\mathcal{M}$ , we will denote the former with bars, e.g.  $\bar{g}$ .

### 3.3.2 Equations of Gauss and Weingarten

Restricting to  $\Sigma^3$ , the first structure equation reads as follows:

$$\begin{aligned} d\bar{\vartheta}^i + \omega^i_j \wedge \bar{\vartheta}^j &= 0 \\ \omega^0_k \wedge \bar{\vartheta}^k &= 0 \end{aligned} \tag{3.23}$$

since  $\vartheta^0$  lies in  $\mathcal{T}^*\Sigma^3$ . Note we have  $\bar{\omega}^i_j = \omega^i_j$  since both satisfy the above equations. That is to say, that the tangential projection of  $\nabla_X Y$  on  $\mathcal{T}\Sigma^3 \subset \mathcal{T}\mathcal{M}$  is equal to that of  $\bar{\nabla}_X Y$ .

The following theorem of Cartan allows us to find  $C^\infty$  functions on  $\Sigma^3$  which relate the connection forms  $\omega^0_i$  and the dual basis  $\vartheta^j$ .

**Theorem 3.3.1.** *If  $\alpha_1, \dots, \alpha_n$  are linearly independent 1-forms on a manifold  $M$  of dimension  $m \geq n$  and  $\beta_1, \dots, \beta_n$  are 1-forms on  $M$  such that*

$$\sum_{i=1}^n \alpha^i \wedge \beta_i = 0 \tag{3.24}$$

then there are smooth symmetric functions  $f_{ij}$  such that

$$\beta = \sum_{j=1}^n f_{ij} \alpha_j. \tag{3.25}$$

*Proof.* See Straumann [17]. □

In light of (3.23), Theorem 3.3.1 guarantees the existence of symmetric smooth functions  $K_{ij}$  such that

$$\omega^0_i = -K_{ij} \bar{\vartheta}^j. \tag{3.26}$$

Worked out in local co-ordinates,  $K_{ij}$  satisfies what are known as the **equations of Weingarten**:

$$\langle \nabla_{e_i} e_j, e_0 \rangle = -\omega^0_j(e_i) = K_{ij} = K_{ji} = -\omega^0_i(e_j) = \langle \nabla_{e_j} e_i, e_0 \rangle. \tag{3.27}$$

Thus we may think of  $K_{ij}$ , known as the **second fundamental form**, or more colloquially, as the **extrinsic curvature**, as a symmetric bilinear form  $K(X, Y)$  which gives the normal components to  $\nabla_X Y$ . As a result, the extrinsic curvature gives a relation between the two connections on  $\Sigma^3$  and  $\mathcal{M}$ . Again, since  $\bar{\omega}^i_j = \omega^i_j$  the extrinsic curvature satisfies the relation:

$$\nabla_X Y - \bar{\nabla}_X Y = -K(X, Y)e_0 \tag{3.28}$$

known as the **equation of Gauss**. For an additional discussion see [17, 18].

We now move on to analyzing the relations between  $\Omega^\mu_\nu$  and  $\bar{\Omega}^i_j$ .

### 3.3.3 Equations of Gauss and Codazzi-Mainardi

Consider the restriction of the second structure equation to  $\mathcal{T}\Sigma^3$ :

$$\begin{aligned} \Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i_0 \wedge \omega^0_j \\ \Omega^0_j &= d\omega^0_j + \omega^0_i \wedge \omega^i_j. \end{aligned} \tag{3.29}$$

Again, using the fact that  $\omega^i_j = \bar{\omega}^i_j$  and  $\omega^0_i = -K_{ij}\bar{\vartheta}^j$  we have

$$\begin{aligned}\Omega^i_j - \bar{\Omega}^i_j &= \omega^i_0 \wedge \omega^0_j \\ &= -K_{ij}\bar{\vartheta}^j \wedge -K_{ij}\bar{\vartheta}^i \\ &= K^i_k K_{jl}\bar{\vartheta}^k \wedge \bar{\vartheta}^l.\end{aligned}\tag{3.30}$$

We see that this relation is just a restatement of (3.28) but in terms of differential forms. Working out the components for  $\Omega^0_j$ :

$$\begin{aligned}\Omega^0_j &= d\omega^0_j + \omega^0_i \wedge \omega^i_j \\ &= -d(K_{ij}\bar{\vartheta}^j) - K_{ik}\bar{\vartheta}^k \wedge \bar{\omega}^i_j \\ &= -dK \wedge \bar{\vartheta}^j - K_{ij}d\bar{\vartheta}^i - K_{ik}\bar{\vartheta}^k \wedge \bar{\omega}^i_j \\ &= \bar{D}K_{ij} \wedge \bar{\vartheta}^i\end{aligned}\tag{3.31}$$

where  $\bar{D}$  is the absolute exterior differential (§7 Appendix). This relation is known as the **equations of Codazzi-Mainardi**.

With these basic equations of Gauss and Codazzi-Mainardi, we are able to list four of the ten (spatial) components of the Ricci and Einstein tensors. In order to find the others, we need to develop the ideas of lapse and shift which we do in the following section following [17].

But first consider the **interior product** (§7) of vectors in the tangent space  $e_\mu$  with the curvature form  $\Omega^\mu_\nu$ :

$$\begin{aligned}\iota_{e_\mu}\Omega^\mu_\nu &= \frac{1}{2}R^\mu_{\nu\sigma\rho}\iota_\mu(\vartheta^\sigma \wedge \vartheta^\rho) \\ &= R_{\nu\rho}\vartheta^\rho.\end{aligned}\tag{3.32}$$

In particular we have (with the use of (3.31))

$$\begin{aligned}R_{0i} &= \Omega^j_0(e_j, e_i) \\ &= d(-K^j_i\bar{\vartheta}^i) + \omega^j_i \wedge (-K^j_k\bar{\vartheta}^k) \\ &= -\bar{D}K^j_i \wedge \bar{\vartheta}^j.\end{aligned}\tag{3.33}$$

Once we introduce the source terms  $T_{0i}$ , the three equations

$$R_{0i} = -\bar{D}K^j_i \wedge \bar{\vartheta}^j = 8\pi T_{0i}\tag{3.34}$$

become known as the **momentum constraints**. Or more commonly written:

$$R_{0i} = \bar{\nabla}_i K^j_j - \bar{\nabla}_j K^j_i = 8\pi T_{0i}.\tag{3.35}$$

Similarly, we use (3.32) and write the first component of the Einstein (trace-reversed Ricci) tensor

$$\begin{aligned}G_{00} &:= R_{00} + \frac{1}{2}R^\mu_\mu \\ &= R_{00} + \frac{1}{2}(-R_{00} + R^i_i) = \frac{1}{2}(R_{00} + R^i_i) \\ &= \frac{1}{2}\left(\Omega^j_0(e_j, e_0) + \Omega^{ji}(e_j, e_i) + \Omega^{0i}(e_0, e_i)\right) \\ &= \frac{1}{2}\Omega^{ij}(e_i, e_j).\end{aligned}\tag{3.36}$$

Re-writting the Codazzi-Mainardi equations (3.30) we have

$$\begin{aligned}
\langle R(e_k, e_l)\vartheta^j, \vartheta^i \rangle &= \Omega^{ij}(e_k, e_l) \\
&= \bar{\Omega}^{ij}(e_k, e_l) + K^{ik}K_{jl}(\bar{\vartheta}^k \wedge \bar{\vartheta}^l)(e_k, e_l) \\
&= \langle \bar{R}(e_k, e_l)\vartheta^j, \vartheta^i \rangle + K(\vartheta^i, \vartheta^k)K(e_j, e_l) - K(e_i, e_l)K(e_j, e_k)
\end{aligned} \tag{3.37}$$

and as a result, we have the **Hamiltonian constraint**:

$$G_{00} = \frac{1}{2} \left( \bar{R} + K^i_i K^j_j - K^i_j K^j_i \right) = 8\pi T_{00}. \tag{3.38}$$

### 3.3.4 Lapse and Shift

We are now ready to introduce the concepts of lapse and shift, which are at the heart of the 3+1 splitting. As we shall see, these functions and vector fields, respectively, describe how to slice space-time into space-like hypersurfaces which can then be interpreted as the initial slice evolving in time.

Let a Lorentzian 4-manifold  $(\mathcal{M}, g_{\mu\nu})$  be given and suppose there is a diffeomorphism

$$\Upsilon : \mathcal{M} \rightarrow \Sigma^3 \times [0, T], \quad T \in \mathbb{R}^+ \tag{3.39}$$

such that the 3-manifolds  $\Sigma_t^3 = \Upsilon^{-1}(\Sigma^3 \times \{t\})$  and the curves  $\Upsilon^{-1}(\{p\} \times [0, T])$  are space-like and time-like, respectively. The latter are known as the **preferred time-like orbits** and their time-like tangent vectors  $\partial_t$  form a vector field on  $\mathcal{M}$ . With respect to each slicing, we can decompose  $\partial_t$  into normal and parallel components:

$$\partial_t := \alpha \mathbf{n} + \bar{\beta} \tag{3.40}$$

where  $\mathbf{n}$  is the (future) directed normal and  $\bar{\beta}$  lies tangent to  $\Sigma_t^3$ . If  $\{\partial_t, \partial_i\}$  is a basis (with  $\langle \mathbf{n}, \partial_i \rangle = 0$ ) for  $\mathcal{T}\mathcal{M}$  and  $\{dt, dx^i\}$  denotes its corresponding orthogonal dual basis for  $\mathcal{T}^*\mathcal{M}$ , we then write  $\bar{\beta} = \beta^i \partial_i$ . Since  $\langle \mathbf{n}, \partial_i \rangle = 0$ ,

$$\begin{aligned}
\langle \partial_t, \partial_t \rangle &= \langle \alpha \mathbf{n} + \beta^i \partial_i, \alpha \mathbf{n} + \beta^i \partial_i \rangle \\
&= \alpha^2 \langle \mathbf{n}, \mathbf{n} \rangle + 2\langle \beta^i \partial_i, \alpha \mathbf{n} \rangle + \langle \beta^i \partial_i, \beta^i \partial_i \rangle \\
&= -(\alpha^2 - \beta^i \beta_i).
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
\langle \partial_t, \partial_i \rangle &= \langle \alpha \mathbf{n} + \beta^j \partial_j, \partial_i \rangle \\
&= \langle \alpha \mathbf{n}, \partial_i \rangle + \langle \beta^j \partial_j, \partial_i \rangle \\
&= \bar{g}_{ij} \beta^j
\end{aligned} \tag{3.42}$$

where  $\bar{g}_{ij} := \langle \partial_i, \partial_j \rangle$  are the components of the induced metric on  $\Sigma_t^3$ . Then we can write the metric  $g_{\mu\nu}$  on  $\mathcal{M}$  as follows:

$$\begin{aligned}
g &= g_{00} dt^2 + g_{0i} dt dx^i + \bar{g}_{ij} dx^i dx^j \\
&= -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\bar{g}_{ij} \beta^j dt dx^i + \bar{g}_{ij} dx^i dx^j \\
&= -\alpha^2 dt^2 + \bar{g}_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt).
\end{aligned} \tag{3.43}$$

From here it follows that the cotangent vectors  $\{dt, dx^i + \beta^i dt\}$  form (another) basis for  $\mathcal{T}^*\mathcal{M}$ .

In the next section, we will compute the first and second structure equations with respect to not this basis, but rather, an orthonormal one. Let  $\{\bar{e}_i\}$  be an orthonormal basis for  $\mathcal{T}\Sigma_t^3$  and let  $\{\bar{\theta}_i\}$  denote the corresponding dual for  $\mathcal{T}^*\Sigma_t^3$ . Then the vectors  $\{e_0 = \mathbf{n}, \bar{e}_i\}$  form a basis for  $\mathcal{T}\mathcal{M}$  and its corresponding dual basis would be  $\{\vartheta^0 = \alpha dt, \vartheta^i = \bar{\theta}^i + \beta^i dt\}$ .



### 3.3.5 Full Dynamical System

Recall the first structure equation is written as follows:

$$0 = d\vartheta^\mu + (\omega^\mu{}_\nu \wedge \vartheta^\nu).$$

We need to work out each of the components showing up on the right-hand side. It is a bit laborious, and we shall work out each of the components one step at a time; we begin with  $d\vartheta^\mu$ . Recall the choice of basis such that  $\bar{\theta}^i = \vartheta^i|_{\mathcal{T}\Sigma_t^3}$ . In particular,

$$\begin{aligned} d(\vartheta^0) = d(\alpha dt) &= \bar{d}\alpha \wedge dt + \alpha \wedge \bar{d}(dt) \\ &= \bar{\nabla}_j \alpha \bar{\theta}^j \wedge \frac{1}{\alpha} \vartheta^0 + 0 \\ &= \frac{1}{\alpha} \bar{\nabla}_j \alpha \vartheta^j \wedge \vartheta^0 \\ &= -\omega^0{}_i \wedge \vartheta^i. \end{aligned} \tag{3.44}$$

Now taking the interior product of  $e_0$  and  $d\theta^0$  we get

$$\iota_{e_0} d\theta^0 = -\omega^0{}_i(e_0)\theta^i = -\frac{1}{\alpha} \bar{\nabla}_i \alpha \theta^i. \tag{3.45}$$

Recall that  $\omega^0{}_i = -K_{ij}$  on  $\Sigma_t^3$  so that (3.45) gives on  $\mathcal{M}$

$$\omega^0{}_i = -K_{ij} + \frac{1}{\alpha} \bar{\nabla}_i \alpha. \tag{3.46}$$

With the use of the first structure equation, we can now find all of the  $d\theta^i$  components as follows: first take the inner product of  $d\theta^i$  with  $e_0$

$$\begin{aligned} \iota_{e_0} d\theta^i &= -\iota_{e_0} [\omega^i{}_0 \wedge \theta^0 + \omega^i{}_j \wedge \theta^j] \\ &= -[K^i{}_j + \omega^i{}_j(e_0)] \theta^j. \end{aligned} \tag{3.47}$$

That is to say,

$$K^i{}_j + \omega^i{}_j(e_0) = -\iota_{\bar{e}_j} \iota_{e_0} d\theta^i. \tag{3.48}$$

Then working on the right-hand side:

$$\begin{aligned} \iota_{\bar{e}_j} \iota_{e_0} d\theta^i &= \iota_{\bar{e}_j} \iota_{e_0} d[\bar{\theta}^i + \beta^i dt] = \iota_{\bar{e}_j} \iota_{e_0} [\bar{d}\bar{\theta}^i + dt \wedge \partial_t \bar{\theta}^i + \bar{d}\beta^i \wedge dt] \\ &= \iota_{\bar{e}_j} \left[ \frac{1}{\alpha} \iota_{\bar{\beta}} (\bar{\omega}^i{}_k \wedge \bar{\theta}^k) + \frac{1}{\alpha} (\partial_t \bar{\theta}^i - \bar{d}\beta^i) \right] \\ &= \frac{1}{\alpha} [(\bar{\omega}^i{}_k \wedge \bar{\theta}^k)(\bar{\beta}, \bar{e}_j) + \partial_t \bar{\theta}^i(\bar{e}_j) - \bar{d}\beta^i(\bar{e}_j)] \\ &= \frac{1}{\alpha} [\bar{\omega}^i{}_j(\bar{\beta}) - \bar{\nabla}_j \beta^i] + \frac{1}{\alpha} \partial_t \bar{\theta}^i(\bar{e}_j). \end{aligned} \tag{3.49}$$

If we set  $\partial_t \bar{\theta}^i = c^i{}_j \bar{\theta}^j$ , we can then decompose (3.46) into symmetric and skew-symmetric components

$$\begin{aligned} \omega_{ij}(e_0) &= -\frac{1}{\alpha} \bar{\omega}_{ij}(\bar{\beta}) + \frac{1}{2\alpha} (\bar{\nabla}_j \beta_i - \bar{\nabla}_i \beta_j) - \frac{1}{2\alpha} (c_{ij} - c_{ji}) \\ K_{ij} &= \frac{1}{2\alpha} [(\bar{\nabla}_j \beta_i + \bar{\nabla}_i \beta_j) - (c_{ij} + c_{ji})] \end{aligned} \tag{3.50}$$

where  $c_{ij} + c_{ji} = (\partial_t \bar{g})_{ij}$  is the tangential derivative of the metric.

In §3.3.3, four of the ten components of the curvature form  $\Omega^\mu{}_\nu$  were calculated. In the last part of this section, we derive the remaining time-dependent components. To that end, we begin with the second structure equations to find

$$\Omega^i{}_0 = d\omega^i{}_0 + \omega^i{}_j \wedge \omega^j{}_0 = -d(K^i{}_j \vartheta^j) + d\left(\frac{1}{\alpha} \bar{\nabla}^i \alpha \vartheta^0\right) + \omega^i{}_j \wedge \left(-K^j{}_i \vartheta^l + \frac{1}{\alpha} \bar{\nabla}^j \vartheta^0\right).$$

To simplify matters, we set out by establishing three important relations. Note that

$$\begin{aligned} -d(K^i{}_j \vartheta^j) &= -dK^i{}_j \wedge \vartheta^j + K^i{}_j d\vartheta^j = -dK^i{}_j \wedge \vartheta^j + K^i{}_j \left(\omega^i{}_l \wedge \vartheta^l - \omega^j{}_0 \wedge \vartheta^0\right) \\ &= -dK^i{}_j \wedge \vartheta^j + K^i{}_j \left(\omega^i{}_l \wedge \vartheta^l - K^j{}_l \vartheta^l \wedge \vartheta^0\right) \end{aligned}$$

and

$$\omega^i{}_j \frac{1}{\alpha} \bar{\nabla}^j \alpha \wedge \vartheta^0 = \omega^i{}_j(\bar{e}_k) \frac{1}{\alpha} \bar{\nabla}^j \alpha \vartheta^k \wedge \vartheta^0$$

and

$$\begin{aligned} d\left(\frac{1}{\alpha} \bar{\nabla}^i \alpha \vartheta^0\right) &= d\left(\frac{1}{\alpha} \bar{\nabla}^i \alpha\right) \wedge \vartheta^0 + \frac{1}{\alpha} \bar{\nabla}^l \alpha d\vartheta^0 \\ &= \left(\frac{1}{\alpha} \bar{\nabla}^i \alpha\right)_{,j} \vartheta^j \wedge \vartheta^0 + \left(\frac{1}{\alpha} \bar{\nabla}^i \alpha \frac{1}{\alpha} \bar{\nabla}^j \alpha\right) \vartheta^i \wedge \vartheta^0 \\ &= \frac{1}{\alpha} \alpha_{,ij} \vartheta^j \wedge \vartheta^0. \end{aligned} \tag{3.51}$$

Combining these we find that

$$\begin{aligned} \Omega^i{}_0 &= -dK^i{}_j \wedge \vartheta^j + K^i{}_j \left(\omega^i{}_l \wedge \vartheta^l - K^j{}_l \vartheta^l \wedge \vartheta^0\right) + \frac{1}{\alpha} \alpha_{,ij} \vartheta^j \wedge \vartheta^0 \\ &\quad + \omega^i{}_j(\bar{e}_k) \frac{1}{\alpha} \bar{\nabla}^j \alpha \vartheta^k \wedge \vartheta^0 - K^j{}_l \omega^i{}_j \wedge \vartheta^l \\ &= \frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha \vartheta^j \wedge \vartheta^0 - dK^i{}_j \wedge \vartheta^j + \left(K^i{}_j \omega^j{}_l - \omega^i{}_j K^j{}_l\right) \wedge \vartheta^l - K^i{}_j K^j{}_l \vartheta^l \wedge \vartheta^0. \end{aligned}$$

Now let  $\bar{\mathcal{L}}_{\bar{\beta}} \bar{g}$  denote the Lie derivative (§7 Appendix) of the metric on  $\Sigma^3_t$ . Then

$$\begin{aligned} (\bar{\mathcal{L}}_{\bar{\beta}} \bar{g})_{ij} &= \bar{\beta}^k \bar{\nabla}_k \bar{g}_{ij} + \left(\bar{\nabla}_j \bar{\beta}^l\right) \bar{g}_{lj} + \left(\bar{\nabla}_i \bar{\beta}^l\right) \bar{g}_{il} \\ &= \bar{\nabla}_i \bar{\beta}_j + \bar{\nabla}_j \bar{\beta}_i. \end{aligned} \tag{3.52}$$

It follows then that (3.50) may be re-written as

$$K = -\frac{1}{2\alpha} (\partial_t - \bar{\mathcal{L}}_{\bar{\beta}}) \bar{g}. \tag{3.53}$$

Again, using (3.32) and (3.52) we have

$$\begin{aligned} R_{00} = \Omega^i{}_0(\bar{e}_i, e_0) &= \frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha \vartheta^j + dK^i{}_i(e_0) - K^i{}_j K^j{}_i \\ &= \frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha \vartheta^j - K^i{}_j K^j{}_i + \frac{1}{\alpha} (\partial_t - \bar{\mathcal{L}}_{\bar{\beta}}) K^i{}_i. \end{aligned} \tag{3.54}$$

Computing the final components of  $R_{ij}$  and using (3.52)

$$\begin{aligned}
R_{ij} &= \Omega^0_i(e_0, \bar{e}_j) + \Omega^k_i(\bar{e}_k, \bar{e}_j) \\
&= -R_{00} + \Omega^k_i(\bar{e}_k, \bar{e}_j) \\
&= -\frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha - dK_{ij}(e_0) + K_{il} (\omega^l_j(e_0) + K^l_j) - \omega^i_l(e_0) K^l_j \\
&\quad + \bar{R}_{ij} + K^i_i K_{ij} - K_{il} K^l_j \\
&= -\frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha - \frac{1}{\alpha} (\partial_t - \bar{\mathcal{L}}_{\bar{\beta}}) K_{ij} + \bar{R}_{ij} + K^i_i K_{ij} + K_{il} \omega^l_j(e_0) + \omega^l_i(e_0) K_{jl} \\
&= -\frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha + \bar{R}_{ij} - 2K_{il} K^l_j - \frac{1}{\alpha} (\partial_t \bar{K} - \bar{\mathcal{L}}_{\bar{\beta}} \bar{K})_{ij} \tag{3.55}
\end{aligned}$$

where a simple calculation shows

$$(\partial_t K - \bar{\mathcal{L}}_{\bar{\beta}} K)_{ij} - (\partial_t - \bar{\mathcal{L}}_{\bar{\beta}}) K_{ij} = -2\alpha K_{il} K^l_j - \alpha (K_{il} \omega^l_j(e_0) + \omega^l_i(e_0) K_{jl}). \tag{3.56}$$

Finally, for completeness, we include all of the ten equations below.

<b>Dynamical components of Ricci tensor</b>	
$R_{00}$	$= \frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha - K^i_j K^j_i + \frac{1}{\alpha} (\partial_t - \bar{\mathcal{L}}_{\bar{\beta}}) K^i_i$
$R_{ij}$	$= \bar{R}_{ij} + K^i_i K_{ij} - 2K_{ik} K^k_j - \frac{1}{\alpha} (\partial_t K - \bar{\mathcal{L}}_{\bar{\beta}} K)_{ij} - \frac{1}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha$
<b>Constraints</b>	
<b>(Momentum)</b>	$R_{0i} = \bar{\nabla}_i K^j_j - \bar{\nabla}_j K^j_i = 8\pi T_{0i}$
<b>(Hamiltonian)</b>	$G_{00} = \frac{1}{2} (R_{00} + R^i_i) = \frac{1}{2} (\bar{R} + K^i_i K^j_j - K^i_j K^j_i) = 8\pi T_{00}$
with	
$\bar{R} = R - K^i_i K^i_i + 2K_{ij} K^{ij} + \frac{2}{\alpha} (\partial_t - \bar{\mathcal{L}}_{\bar{\beta}}) K^i_i + \frac{2}{\alpha} \bar{\nabla}_j \bar{\nabla}^i \alpha.$	

Our derivation followed [17]; for alternative discussions see [18, 2].

## Chapter 4

# The Hamiltonian and Momentum Constraints

Consider a general space-like hypersurface  $\Sigma^3 \hookrightarrow \mathcal{M}$ . We assume this “slice” of space-time is compact and connected with boundary. Given the induced Riemannian metric  $\bar{g}_{ij}$ , the extrinsic curvature symmetric two-tensor  $K_{ij}$ , and energy source terms  $\mathbf{j}$  and  $\rho$  satisfying  $-\rho^2 + \mathbf{j}^2 \leq 0$ , one says the initial data  $(\bar{g}_{ij}, K_{ij}, \rho, \mathbf{j}^j)$  generates a spacetime solution on  $\Sigma^3 \times \mathbb{R}$  if and only if they satisfy both the Hamiltonian and momentum constraints. That is, if they satisfy

$$\begin{aligned}\bar{R} + K^{ij}K_{ij} - \tau^2 &= 16\pi\rho \\ \bar{\nabla}_j(K^{ij} - \bar{g}^{ij}\tau) &= 8\pi j^j.\end{aligned}\tag{4.1}$$

where  $\bar{R}$  is the Ricci scalar,  $\bar{\nabla}_i$  is the covariant derivative built from the metric  $\bar{g}_{ij}$  and  $\tau$  is the trace (with respect to  $\bar{g}_{ij}$ ) of the symmetric two-tensor  $K^{ij}$ .

As is, these tensor equations form an underdetermined system. Introducing a semi-decoupling decomposition of the constraints, first introduced in [12, 13, 14], one transforms the problem into a determined elliptic system where the unknowns are now a positive definite scalar function  $\phi$  and vector field  $\mathbf{w}$ .

### 4.1 York Decomposition

Suppose  $\bar{g}_{ij}$  and  $\hat{g}_{ij}$  are 3-metrics on  $\Sigma^3$  and there is a transformation such that

$$\bar{g}_{ij} \mapsto e^f \hat{g}_{ij}\tag{4.2}$$

for some scalar positive definite function  $f$ ; we then say that  $\hat{g}_{ij}$  is **conformal** to  $\bar{g}_{ij}$ . In the future, all hatted quantities, e.g.  $\hat{R}$ , will be with respect to  $\hat{g}$ . While there is no particular rhyme or reason for the choice of  $f$ , in the decomposition undertaken by York the selection  $f = \ln \phi^4$  is made, so that  $\phi > 0$ . That is, York supposes the two metrics are related in such a way that we have

$$\bar{g}_{ij} = \phi^4 \hat{g}_{ij}.\tag{4.3}$$

The choice of conformal factor reduces the six degrees of freedom present in  $\bar{g}_{ij}$  to just one:  $\phi$ . This is how the York decomposition transforms the underdetermined tensor equations into a determined system.

In order to write the system in elliptic form, the first order of business is to be able to relate the new conformal Ricci scalar and conformal extrinsic curvature to those built from the metric  $\bar{g}_{ij}$ .

**Proposition 4.1.1.** *Let  $\bar{R}$ ,  $\hat{R}$  be the Ricci scalars built from the metrics  $\bar{g}_{ij}$  and  $\hat{g}_{ij}$ , respectively. Then,*

$$\bar{R} = \phi^{-5}(\phi\hat{R} - 8\hat{\Delta}\phi) \quad (4.4)$$

where  $\hat{\Delta}$  denotes the Laplacian with respect to the conformal metric.

*Proof.* See Wald [18].

**Proposition 4.1.2.** *Let  $\bar{T}^{ij}$  and  $\hat{T}^{ij}$  be two symmetric and trace-less tensors whose respective metrics are conformally related by  $e^f$ . Then*

$$\bar{T}^{ij} = e^{-(\frac{n+2}{2})f}\hat{T}^{ij} \quad (4.5)$$

and

$$\nabla T^{ij} = e^{-(\frac{n+2}{2})f}\hat{\nabla}\hat{T}^{ij}. \quad (4.6)$$

*Proof.* See Straumann [17].

In particular for the York Decomposition,  $e^{-(\frac{n+2}{2})f} = \phi^{-10}$ .

To begin the decomposition we decompose the extrinsic curvature into a trace-less part  $\sigma^{ij}$  and a trace part  $\tau$ . It is not hard to see that by taking traces

$$K^{ij} = \bar{\sigma}^{ij} + \frac{1}{3}\bar{g}^{ij}\tau. \quad (4.7)$$

Now note by Proposition 4.1.2,  $\bar{\sigma}^{ij} = \phi^{-10}\hat{\sigma}^{ij}$  and since  $\bar{\sigma}_{ij} = \bar{g}_{ik}\bar{g}_{jl}\bar{\sigma}^{kl}$ ,

$$\begin{aligned} \bar{\sigma}_{ij} &= (\phi^4\hat{g}_{ik})(\phi^4\hat{g}_{jl})(\phi^{-10}\hat{\sigma}^{kl}) \\ &= \phi^{-2}\hat{\sigma}_{ij}. \end{aligned} \quad (4.8)$$

With the use of Proposition 4.1.1 and (4.7) and (4.8) the Hamiltonian constraint reads

$$\begin{aligned} 0 &= \bar{R} + K^{ij}K_{ij} - \tau^2 - 16\pi\rho \\ &= \phi^{-5}(\phi\hat{R} - 8\hat{\Delta}\phi) + (\bar{\sigma}^{ij} + \frac{1}{3}\bar{g}^{ij}\tau)(\bar{\sigma}_{ij} + \frac{1}{3}\bar{g}_{ij}\tau) - \tau^2 - 16\pi\rho \\ &= \phi^{-5}(\phi\hat{R} - 8\hat{\Delta}\phi) + \bar{\sigma}^{ij}\bar{\sigma}_{ij} + \frac{2}{3}\bar{g}^{ij}\bar{\sigma}_{ij} + \frac{1}{9}\bar{g}^{ij}\bar{g}_{ij}\tau^2 - \tau^2 - 16\pi\rho \\ &= \phi^{-5}(\phi\hat{R} - 8\hat{\Delta}\phi) + (\phi^{-10}\hat{\sigma}^{ij})(\phi^{-2}\hat{\sigma}_{ij}) - \frac{2}{3}\tau^2 - 16\pi\rho \\ &= \phi^{-5}(\phi\hat{R} - 8\hat{\Delta}\phi) + \phi^{-12}\hat{\sigma}^{ij}\hat{\sigma}_{ij} - \frac{2}{3}\tau^2 - 16\pi\rho \end{aligned} \quad (4.9)$$

where we used the fact that  $\bar{\sigma}$  is trace-free in the last line, i.e. that  $\bar{g}^{ij}\bar{\sigma}_{ij} = 0$ . Now multiplying through  $\phi^5$  we arrive at

$$\phi\hat{R} - 8\hat{\Delta}\phi + \phi^{-7}\hat{\sigma}^{ij}\hat{\sigma}_{ij} - \frac{2}{3}\phi^5\tau^2 = 16\pi\rho\phi^5. \quad (4.10)$$

Similar to how we can decompose a vector into a divergence-free part and a curl-free part, we decompose our symmetric trace-free tensor  $\hat{\sigma}^{ij}$  into a divergence-free part  $\hat{\sigma}_*^{ij}$  and

a ‘‘longitudinal’’ part  $\hat{\mathcal{L}}\mathbf{w}$ , where  $\mathbf{w}$  is some vector field on  $\Sigma^3$  and  $\hat{\mathcal{L}}$  is the conformal Killing operator in three-dimensions:

$$(\hat{\mathcal{L}}\mathbf{w})^{ij} = \hat{\nabla}^i \mathbf{w}^j + \hat{\nabla}^j \mathbf{w}^i - \frac{2}{3} \hat{g}^{ij} \hat{\nabla}_k \mathbf{w}^k. \quad (4.11)$$

Thus we can write (4.10) as

$$\phi \hat{R} - 8 \hat{\Delta} \phi + \phi^{-7} [\hat{\sigma}_* + (\hat{\mathcal{L}}\mathbf{w})]^2 - \frac{2}{3} \phi^5 \tau^2 = 16\pi \rho \phi^5 \quad (4.12)$$

where we use the notation  $T^2 := T^{ij} T_{ij}$ . We follow a similar procedure for the momentum constraint. Begin by writing

$$\begin{aligned} K^{ij} - \bar{g}^{ij} \tau &= (\bar{\sigma}^{ij} + \frac{1}{3} \bar{g}^{ij} \tau) - \bar{g}^{ij} \tau \\ &= \phi^{-10} \hat{\sigma}_*^{ij} + \phi^{-10} (\hat{\mathcal{L}}\mathbf{w})^{ij} - \frac{2}{3} \phi^{-4} \bar{g}^{ij} \tau. \end{aligned}$$

Then applying the covariant derivative and using Proposition 4.1.2 we have

$$\begin{aligned} \bar{\nabla}_j (K^{ij} - \bar{g}^{ij} \tau) &= \phi^{-10} \hat{\nabla}_j \hat{\sigma}_*^{ij} + \phi^{-10} \hat{\nabla}_j (\hat{\mathcal{L}}\mathbf{w})^{ij} - \frac{2}{3} \phi^{-4} \bar{g}^{ij} \bar{\nabla}_j \tau \\ &= \phi^{-10} \hat{\nabla}_j (\hat{\mathcal{L}}\mathbf{w})^{ij} - \frac{2}{3} \phi^{-4} \bar{g}^{ij} \bar{\nabla}_j \tau \end{aligned} \quad (4.13)$$

where in the last line we used the fact that  $\hat{\sigma}_*^{ij}$  is divergence-free, i.e. that  $\hat{\nabla}_j \hat{\sigma}_*^{ij} = 0$ . After multiplying through  $\phi^{-10}$ , one reads the momentum constraint as

$$\hat{\nabla}_j (\hat{\mathcal{L}}\mathbf{w})^{ij} = \frac{2}{3} \phi^6 \bar{g}^{ij} \bar{\nabla}_j \tau + 8\pi j^j \phi^{10}. \quad (4.14)$$

We are almost complete with the decomposition but there is still the issue of how to conformally vary the source terms  $\mathbf{j}$  and  $\rho$ . This is really vital because we want to ensure that the solutions to the elliptic system are *physically realistic*. If the choice  $j^j = \phi^{-10} \hat{j}^j$  is made, the momentum constraint completely decouples when  $\tau$  is constant. Note then

$$\begin{aligned} -\rho^2 + \bar{g}_{ij} \hat{j}^i \hat{j}^j &\leq 0 \\ -\rho^2 + (\hat{g}_{ij} \phi^4) (\phi^{-10} \hat{j}^i) (\phi^{-10} \hat{j}^j) &\leq 0 \\ -\rho^2 + \phi^{-16} \hat{g}_{ij} \hat{j}^i \hat{j}^j &\leq 0. \end{aligned} \quad (4.15)$$

Thus the choice  $\rho = \phi^{-8} \hat{\rho}$  is a natural one, for then

$$\phi^{-16} [-\hat{\rho}^2 + \hat{g}_{ij} \hat{j}^i \hat{j}^j] \leq 0. \quad (4.16)$$

Suffice it to say, we put

$$\begin{aligned} \rho &= \phi^{-8} \hat{\rho} \\ j^j &= \phi^{-10} \hat{j}^j \end{aligned}$$

to guarantee the solutions are physical.

From here we arrive at the final decomposition of the Hamiltonian and momentum constraints:

$$-\hat{\Delta} \phi + \frac{\hat{R}}{8} \phi + \frac{1}{12} \tau \phi^5 - \frac{1}{8} [\hat{\sigma}_* + (\hat{\mathcal{L}}\mathbf{w})]^2 \phi^{-7} - 2\pi \hat{\rho} \phi^{-3} = 0$$

$$\hat{\nabla}_b(\hat{\mathcal{L}}\mathbf{w})^{ij} = \frac{2}{3}\phi^6\bar{g}^{ij}\bar{\nabla}_j\tau + 8\pi\hat{j}^j. \quad (4.17)$$

So by treating  $\{\hat{g}_{ij}, \tau, \hat{\sigma}_*^{ij}, \hat{\rho}, \hat{j}^j\}$  as initial data and  $\{\phi, \mathbf{w}\}$  as a solution to the elliptic constraints, one can recover the physical initial data by setting

$$\begin{aligned} \bar{g}_{ij} &= \phi^4\hat{g}_{ij} \\ K^{ij} &= \phi^{-10}[\hat{\sigma}_*^{ij} + (\hat{\mathcal{L}}\mathbf{w})^{ij}] + \frac{1}{3}\phi^{-4}\bar{g}^{ij}\tau \\ \rho &= \phi^{-8}\hat{\rho} \\ j^j &= \phi^{-10}\hat{j}^j. \end{aligned} \quad (4.18)$$

In order to make further discussions on the coupled system easier, we wish to employ the use of simpler notation. Define the action of the Hamiltonian operator  $L$  by  $L\phi := -\hat{\Delta}\phi$  and the action of the momentum operator  $\mathbb{L}$  by  $\mathbb{L}\mathbf{w} := -\hat{\nabla} \cdot (\hat{\mathcal{L}}\mathbf{w})$ . Put

$$f(\phi, \mathbf{w}) := a_\tau\phi^5 + a_R - a_\rho\phi^{-3} - a_{\mathbf{w}}\phi^{-7} \quad (4.19)$$

$$\mathbf{f}(\phi) := \mathbf{b}_\tau\phi^6 + 8\pi\mathbf{j} \quad (4.20)$$

where

$$a_R := \frac{\hat{R}}{8}, \quad a_\tau := \frac{\tau^2}{12}, \quad a_\rho := 2\pi\hat{\rho}, \quad a_{\mathbf{w}} := \frac{1}{8}[\hat{\sigma}_* + (\hat{\mathcal{L}}\mathbf{w})]^2, \quad \mathbf{b}_\tau := \frac{2}{3}\bar{g}\bar{\nabla}\tau. \quad (4.21)$$

Then we may write the decomposition of the Einstein constraint equations as

$$\begin{aligned} L\phi + f(\phi, \mathbf{w}) &= 0 \\ \mathbb{L}\mathbf{w} + \mathbf{f}(\phi) &= 0. \end{aligned} \quad (4.22)$$

## 4.2 Dirichlet-Robin Boundary Value Problem

Since our intent is to work on manifolds with boundary, the statement of the problem in (4.22) is incomplete. That is to say, we must state the constraints as a Dirichlet-Robin boundary value problem. Although this is quite a delicate matter, for the solvability of the constraints rests on the boundary condition choice, we will take the Dirichlet-Robin boundary value problem for the York decomposed constraint equations as the following:

**Problem 4.2.1.** Given the freely specifiable smooth functions  $\tau$ ,  $\hat{\sigma}_*^{ij}$ ,  $\hat{\rho}$ , and  $\hat{\mathbf{j}}$  in  $\Omega$  with  $-\hat{\rho}^2 + \hat{\mathbf{j}}^2 \leq 0$  and the smooth Dirichlet boundary data  $\hat{\phi}_D$  on  $\partial\Omega_D$  and  $\hat{\mathbf{w}}_D$  on  $\partial\Omega_D$ , and smooth Robin boundary data  $\hat{\phi}_N$ ,  $K$  on  $\partial\Omega_N$  and  $\hat{\mathbf{w}}_N$ ,  $\mathbb{K}$  on  $\partial\Omega_N$ , find a scalar field  $\phi$  and a vector field  $\mathbf{w}$  in  $\Omega$  solution of the system

$$L\phi + f(\phi, \mathbf{w}) = 0 \text{ in } \Omega, \quad \phi = \hat{\phi}_D \text{ on } \partial\Omega_D, \quad \mathbf{n} \cdot \nabla\phi + K\phi = \hat{\phi}_N \text{ on } \partial\Omega_N \quad (4.23)$$

$$\mathbb{L}\mathbf{w} + \mathbf{f}(\phi) = 0 \text{ in } \Omega, \quad \mathbf{w} = \hat{\mathbf{w}}_D \text{ on } \partial\Omega_D, \quad \mathbf{n} \cdot \nabla\mathbf{w} + \mathbb{K}\mathbf{w} = \hat{\mathbf{w}}_N \text{ on } \partial\Omega_N. \quad (4.24)$$

# Chapter 5

## Previous Solution Techniques

Over the years, much work has been done to prove results for the individual constraint equations and the coupled system. Much of the motivation for this thesis and the collaboration which ensued from M. Holst and G. Nagy arose as a result of the work of J. Isenberg and collaborators over the past decade or so, as well as the collection of notes by M. Holst [7].

### 5.1 J. Isenberg (1995)

Although Isenberg [9] does not show existence of solutions to the coupled system, he does establish a number of critical results for the Hamiltonian constraint. This constraint, commonly referred to as the **Lichnerowicz equation**, can be isolated from the coupled system by assuming  $\tau$  is constant. In doing so,  $\nabla\tau$  vanishes and the system decouples, leaving:

$$-\hat{\Delta}\phi + \frac{\hat{R}}{8}\phi + \frac{1}{12}\tau\phi^5 - \frac{1}{8}[\hat{\sigma}_* + (\hat{\mathcal{L}}\mathbf{w})]^2\phi^{-7} - 2\pi\hat{\rho}\phi^{-3} = 0. \quad (5.1)$$

Isenberg uses a common sub- and super-solution PDE technique to establish existence (and uniqueness) of solutions. We shall state his main theorem for completeness.

**Theorem 5.1.1.** *Let  $(\Sigma^3, \bar{g}_{ij})$  be a closed Riemannian 3-manifold without boundary with  $\bar{g}_{ij} \in C^3(\Sigma^3)$ . Suppose  $\hat{\sigma}_*^{ij} \in W^{2,p}(\Sigma^3)$ , for  $p > 3$ , and  $\tau$  is constant. Then (5.1) has or does not have a positive definite solution  $\phi \in C^{2,\alpha}(\Sigma^3)$ ,  $\alpha \in (0, 1 - 3/p)$ , based upon the following table:*

	$\tau = 0$	$\tau \neq 0$
$\hat{\sigma}_*^{ij} \equiv 0$	$(N, Y^\dagger, N)$	$(N, N, Y)$
$\hat{\sigma}_*^{ij} \not\equiv 0$	$(Y, N, N)$	$(Y, Y, Y)$

where  $(\cdot, \cdot, \cdot)$  denotes either a Yes or a No to the existence of a solution depending upon the signature value of the Ricci scalar  $R$  on all of  $\Sigma^3$  :  $(+, 0, -)$ .

†: any constant is a solution; in all other cases, when a solution exists, it is unique.

### 5.2 J. Isenberg and V. Moncrief (1996)

In the following year, Isenberg teamed up with V. Moncrief and published a result [10] which took a step forward in solving the coupled system. Below is the statement of their main theorem:



**Theorem 5.2.1.** *Let  $\Sigma^3$  be a compact 3-manifold without boundary. Suppose  $\bar{g}_{ij}$  is a  $C^3$  metric which has Ricci Scalar  $-1$  and no conformal Killing vector fields. Furthermore, let  $\hat{\sigma}_*^{ij} \in W^{2,p}(\Sigma^3)$ ,  $p > 3$ . If for  $\tau \in W^{1,p}(\Sigma^3)$  non-zero with*

$$C \frac{\max |\nabla \tau|^2}{\min \tau^2} < 1 \quad (5.2)$$

and

$$\Lambda / \Theta < 1 \quad (5.3)$$

where

$$\Lambda = \frac{1}{8} \left( 5 \frac{\min \tau^2}{\max \tau^2} - 1 \right) \quad (5.4)$$

and

$$\Theta = C (\max |\sigma_*^{ij}| + C_1 (\phi_\infty^+)^6 \max |\nabla \tau|) (\phi_\infty^+)^5 (\phi_\infty^-)^{-7} \max |\nabla \tau| \quad (5.5)$$

for some constants  $C, C_1 > 0$  and global sub- and super-solutions to the Hamiltonian constraint  $\phi_\infty^-$  and  $\phi_\infty^+$ , respectively, then there exists a unique solution to (4.22) with  $\phi \in C^{2,\alpha}(\Sigma^3)$  and  $\mathbf{w} \in C^{3,\alpha}(\Sigma)$  for  $\alpha = 1 - 3/p$ .

Much of the work in the paper is to establish the existence of these global sub- and super-solutions  $\phi_\infty^-$  and  $\phi_\infty^+$ . In order to do so, these growth conditions on  $\tau$  were needed. As a result, we are left with conditions on  $\tau$  which are more or less similar to the assumption that  $\nabla \tau = 0$ .

Analyzing these results, we begin to question whether this condition has physical meaning, or if it is just a limitation of the proof technique. Since  $\tau$  is the extrinsic curvature of the foliation, it is imaginable there are places in the Universe where curvature is large and still the constraints are valid. Case in point, for Minkowski space-time just choose a foliation with high enough  $|\nabla \tau|$ . Therefore, it is reasonable to suspect that the condition on  $\tau$  is somewhat artificial, and just a by-product of the techniques employed.

### 5.3 D. Maxwell (2005)

More recently, we have the results of D. Maxwell [15]. The assumption that  $\tau$  is constant is made and the set-up is similar to [9]. However, a greater class of rough solutions is studied by working in  $W^{k,2}(\mathcal{M})$  for  $k > 3/2$ . The paper includes a few results; we include the main result below.

**Theorem 5.3.1.** *Suppose  $(\mathcal{M}^n, g)$  is a compact Riemannian  $n$ -manifold with  $g \in W^{k,2}(\mathcal{M}^n)$ ,  $k > n/2$ . Suppose that  $\tau$  is constant and  $\hat{\sigma}_* \in W^{k-1,2}(\mathcal{M}^n)$ . Then there exists a positive solution  $\phi \in W^{k,2}(\mathcal{M}^n)$  to the Lichnerowicz equation (5.1)  $\iff$  either one of the following conditions holds:*

- i.  $\hat{\sigma}_* \neq 0$  and  $\tau \neq 0$*
- ii.  $R > 0$ ,  $\hat{\sigma}_* \neq 0$  and  $\tau = 0$*
- iii.  $R = 0$ ,  $\hat{\sigma}_* = 0$  and  $\tau = 0$*
- iv.  $R < 0$ ,  $\hat{\sigma}_* = 0$  and  $\tau \neq 0$*

where  $R$  is the Ricci scalar of the metric  $g$ . In cases (i) (ii) and (iv) the solution is unique; in (iii) the solution is unique up to a multiplicative positive constant.

Isenberg and Moncrief use a variant of a contraction mapping argument to establish the existence and uniqueness of solutions to the coupled system for a particular Yamabe subclass. In the work done with M. Holst and G. Nagy [8], more general fixed point arguments are utilized which cover a broader set of Yamabe classes, and allow for rough data and weak solutions. While the condition on  $\tau$  is weakened slightly, we have to be content with the fact that using fixed point arguments seems to have limitations. Although we will not discuss the alternative fixed point arguments employed here, we will discuss the strong differentiability assumptions made in [10] and weaken them considerably.

In this thesis we will develop some results on the Hamiltonian constraint which can be used to analyze the coupled system. While the set-up will be similar to Isenberg's work, we will make no assumption on  $\tau$ , except that  $\tau \neq 0$ . The results are interesting in their own right because we will be able to weaken the assumptions made in [9] by working in  $W^{1,2} \cap L^\infty$ . We skirt any of the strong differentiability assumptions by using weak maximum principles to take advantage of the Banach algebra structure present in  $L^\infty$ .

## Chapter 6

# Weak Formulation of Hamiltonian Constraint

At this stage, we must qualify what we mean by a *solution* to the Hamiltonian constraint. If we mean an **classical** solution to

$$-\hat{\Delta}\phi + \frac{\hat{R}}{8}\phi + \frac{1}{12}\tau\phi^5 - \frac{1}{8}[\hat{\sigma}_* + (\hat{\mathcal{L}}\mathbf{w})]^2\phi^{-7} - 2\pi\hat{\rho}\phi^{-3} = 0 \quad (6.1)$$

we have to insist that the solution is *at least* twice differentiable so that  $\hat{\Delta}\phi$  even makes sense. As with many PDEs modeling physical phenomena, sometimes this is requiring too much; indeed, there are techniques which allow one to show the existence of less differentiable solutions. We call these solutions **weak**. In the next section, we will formalize the weak problem, and define what a weak solution is. It will become clear that every classical solution is a weak solution. Thus a common method in establishing solutions to PDEs is to split the work in two. First, establish the existence of weak solutions, then work on showing that it is *regular* enough—that is, it is smooth enough to indeed be a classical solution.

Suppose that  $\phi$  is a classical solution. Then recall we wrote the Hamiltonian constraint as  $L\phi + f(\phi, \mathbf{w}) = 0$ . This equation remains valid if we multiply by a smooth (test) function  $\psi$  and integrate over  $\Omega$ :

$$(L\phi, \psi) + (f(\phi, \mathbf{w}), \psi) = 0 \quad (6.2)$$

where  $(\cdot, \cdot)$  is the usual  $L^2(\Omega)$  inner product. To simplify  $(L\phi, \psi) = (-\hat{\Delta}\phi, \psi)$  we use integration by parts to get **Green's First Identity**:

$$(-\hat{\Delta}\phi, \psi) = (\hat{\nabla}\phi, \hat{\nabla}\psi) - (\text{tr}_N(\mathbf{n} \cdot \hat{\nabla}\phi), \text{tr}_N\psi)_N - (\text{tr}_D(\mathbf{n} \cdot \hat{\nabla}\phi), \text{tr}_D\psi)_D \quad (6.3)$$

where  $(\cdot, \cdot)_N, (\cdot, \cdot)_D$  is the  $L^2(\partial\Omega)$  inner product on the boundary of  $\Omega$ . Note since we are able to move some of the differentiability on  $\phi$  to  $\psi$ , it is reasonable to begin looking for (weak) solutions in a space where one derivative is well behaved under the  $L^2$  norm—that is, in  $W^{1,2}(\Omega)$ . In fact, it is sufficient to take  $W^{1,2}(\Omega)$  to be both the test space and the solution space. However, generally the space of test functions is not taken to be all of  $W^{1,2}(\Omega)$  but rather the subspace

$$W_D^{1,2}(\Omega) := \{\phi \in W^{1,2}(\Omega) : \text{tr}_D\phi = 0\}. \quad (6.4)$$

That is, functions in  $W^{1,2}(\Omega)$  which vanish on the Dirichlet boundary. As a result, Green's identity simplifies since the last term on the right vanishes.

## 6.1 Weak Dirichlet-Robin Boundary Value Problem

We follow the weak formulation introduced in [6, 7] and also in [8]. Introduce the bilinear form

$$a_L : W^{1,2}(\Omega) \times W_D^{1,2}(\Omega) \rightarrow \mathbb{R} \quad (6.5)$$

with action defined by

$$a_L(\phi, \psi) := (\hat{\nabla} \phi, \hat{\nabla} \psi) + (K \operatorname{tr}_N \phi, \operatorname{tr}_N \psi)_N \quad (6.6)$$

where the Robin coefficient  $K \in L^\infty(\partial\Omega_N)$  satisfies the following for some real number  $\hat{k}$  and all  $\phi \in W^{1,2}(\Omega)$ :

$$\hat{k} \|\operatorname{tr}_N \phi\|_N^2 \leq (K \operatorname{tr}_N \phi, \operatorname{tr}_N \phi)_N, \quad (6.7)$$

with  $\|\cdot\|_N$  being the  $L^2$  norm on the boundary  $\partial\Omega_N$ . Now fix the source terms in the following spaces:

$$\tau \in L^{12/5}(\Omega), \quad \hat{\rho} \in L^{6/5}(\Omega), \quad \hat{\sigma}_*^{ij} \in L^{12/5}(\Omega), \quad \hat{\mathbf{j}} \in L^{12/5}(\Omega), \quad \mathbf{w} \in W^{1,12/5}(\Omega).$$

Now given any two functions  $\phi_1, \phi_2 \in L^\infty(\Omega)$  with  $\phi_1 \leq \phi_2$  a.e. in  $\Omega$ , put

$$[\phi_1, \phi_2] := \{\phi \in L^\infty(\Omega) : \phi_1 \leq \phi \leq \phi_2\}. \quad (6.8)$$

Suppose  $0 < \phi_1 \leq \phi_2$ , then define the operator

$$F_f : [\phi_1, \phi_2] \cap W^{1,2}(\Omega) \times W^{1,\infty}(\Omega) \rightarrow [W^{1,2}(\Omega) \times W^{1,\infty}(\Omega)]^* \quad (6.9)$$

with action

$$F_f(\phi, \mathbf{w})(\psi) := (f(\phi, \mathbf{w}), \psi) - (\hat{\phi}_N, \operatorname{tr}_N \psi)_N \quad (6.10)$$

for all  $\psi \in W_D^{1,2}(\Omega)$ .

Now recall the properties of the trace operator  $\operatorname{tr}_D$ . In particular, given an element  $\hat{\phi}_D \in W^{1/2,2}(\partial\Omega_D)$ , there exists an element  $\phi_0 \in W^{1,2}(\Omega)$  such that  $\operatorname{tr}_D \phi_0 = \hat{\phi}_D$ . The element  $\phi_0$  is called the **extension** of  $\hat{\phi}_D$ .

We may now state the weak boundary value problem for the Hamiltonian constraint. Fix Dirichlet and Robin boundary data

$$\hat{\phi}_D \in W^{1/2,2}(\partial\Omega_D), \quad (\hat{\phi}_N, \cdot) \in [W^{1,2}(\partial\Omega_N)]^*$$

with the extension of  $\hat{\phi}_D$  being  $\phi_0 \in [\phi_1, \phi_2] \cap W^{1,2}(\Omega)$ , then find an element  $\phi \in \phi_0 + W_D^{1,2}(\Omega)$  with a solution to

$$\mathbb{H}(\phi)(\psi) := a_L(\phi, \psi) + F_f(\phi, \mathbf{w})(\psi) = 0 \quad (6.11)$$

for all  $\psi \in W_D^{1,2}(\Omega)$ . Such a  $\phi$  is a **weak solution** to the Hamiltonian constraint.

In this section, we only concern ourselves with the existence of weak solutions; that is, we will not be addressing any regularity. To that end, we begin first by establishing *a priori* bounds weak solutions must satisfy.

## 6.2 A Priori w-Dependent $L^\infty$ -Estimates

In this section we establish a priori  $L^\infty$  estimates for solutions to the Hamiltonian constraint. These results are not only critical to fix point arguments for the coupled system, but are also of fundamental importance for developing error estimates for numerical methods.

Consider the non-linear function

$$Q(x) := ax^5 + bx - cx^{-3} - dx^{-7} \quad (6.12)$$

with  $b \in \mathbb{R}$ ,  $a \in \mathbb{R}^+$  and  $c, d \in \mathbb{R}_{\geq 0}$ . As we shall see shortly, the number of positive roots of  $Q$  will be of interest to us. Since  $x = 0$  is not a solution to  $Q(x) = 0$  we can deduce that

$$Q(x) = 0 \iff Q(x)x^7 = 0, \quad x > 0. \quad (6.13)$$

So put  $P(x) := Q(x)x^7 = ax^{12} + bx^8 - cx^4 - d$ . It follows that  $P(x) = 0$ , and hence  $Q(x) = 0$ , has at most 12 (complex) solutions. Since we are interested in only real, positive solutions, a result from Descartes allows us to put a sharper bound on the number of possible solutions.

**Lemma 6.2.1.** (*Descartes's Rule of Signs*) *Let  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  be a polynomial over the reals. Then the number of positive real roots is either equal to the number of sign changes between consecutive nonzero coefficients or less than it by a multiple of two.*

A discussion of this result can be found in [5].

**Proposition 6.2.1.** *Consider the polynomial  $P(x) = ax^{12} + bx^8 - cx^4 - d$ . Suppose  $a \in \mathbb{R}^+$ . Then when either i.)  $b < 0$  and  $c, d \in \mathbb{R}_{\geq 0}$  or ii.)  $b \geq 0$  and  $c, d \in \mathbb{R}_{\geq 0}$  with at least one being non-zero,  $P(x)$  has a unique positive root.*

*Proof.* Follows immediately from Lemma 6.2.1. Just note the assumptions are precisely such that one sign change occurs between consecutive coefficients. Thus  $P$  has at most one positive root. Moreover, by the Intermediate Value Theorem we have the existence of at least one positive root since  $a > 0$  and  $P(0) = -d \leq 0$ . The result follows.  $\square$

**Assumption 6.2.1.** At this time, we will insist that the data appearing in the Hamiltonian constraint satisfies the following:

$$\hat{g}_{ij}, \tau, \mathbf{w} \in W^{1,\infty}(\Sigma^3)$$

$$\hat{\rho}, \hat{\sigma}_*^{ij} \in L^\infty(\Sigma^3).$$

These assumptions here are *more* regular than is needed for the existence proof; c.p. Theorem 6.3.2. We need them in order to make meaning out of the infimum and supremum of functions over  $\Omega$ . However, weaker regularity conditions on the coefficients are developed in [8] that still allow for establishing a priori  $L^\infty$  bounds.

**Assumption 6.2.2.** Furthermore, if  $a_R < 0$ , then suppose  $\inf a_\tau \neq 0$ , else if  $a_R \geq 0$ , then suppose  $\inf a_w \neq 0$ .

The latter assumption allows us to consider the following polynomials, all of which have a unique real positive root by Proposition 6.2.1:

$$\begin{aligned}
g_+(\phi, \mathbf{w}) &:= \sup a_\tau \phi^{12} + \sup a_R \phi^8 - \inf a_\rho \phi^4 - \inf a_{\mathbf{w}} \\
g_-(\phi, \mathbf{w}) &:= \sup a_\tau \phi^{12} - \inf |a_R| \phi^8 - \inf a_\rho \phi^4 - \inf a_{\mathbf{w}} \\
h_+(\phi, \mathbf{w}) &:= \inf a_\tau \phi^{12} + \inf a_R \phi^8 - \sup a_\rho \phi^4 - \sup a_{\mathbf{w}} \\
h_-(\phi, \mathbf{w}) &:= \inf a_\tau \phi^{12} - \sup |a_R| \phi^8 - \sup a_\rho \phi^4 - \sup a_{\mathbf{w}}
\end{aligned}$$

We will call  $\alpha_{\mathbf{w}}$  the root of either  $g_+(\phi, \mathbf{w})$  or  $g_-(\phi, \mathbf{w})$  and  $\beta_{\mathbf{w}}$  the root of either  $h_+(\phi, \mathbf{w})$  or  $h_-(\phi, \mathbf{w})$ . For our concerns here, we loose nothing if we assume that  $0 < \alpha_{\mathbf{w}} \leq \phi_0 \leq \beta_{\mathbf{w}} < +\infty$ , where  $\phi_0$  is the extension of the Dirichlet data to the boundary.

In this notation we can state one of our main theorems. This result generalizes the result in [7] to non-monotone non-linearities in the Hamiltonian constraint. See also [8] for an even more general version of this result.

**Theorem 6.2.1.** *Put  $\phi^+ := (\phi - \beta_{\mathbf{w}})^+$  and  $\phi^- := (\phi - \alpha_{\mathbf{w}})^-$ . Suppose the Robin boundary data  $(\hat{\phi}_N, \cdot) \in [W^{1/2,2}(\partial\Omega_N)]^*$  satisfies*

$$(K \text{tr}_N \phi - \hat{\phi}_N, \text{tr}_N \phi)_N \geq 0.$$

*If  $\phi \in \phi_0 + W_D^{1,2}(\Sigma^3)$  is a weak solution to the Hamiltonian constraint, then*

$$0 < \alpha_{\mathbf{w}} \leq \phi \leq \beta_{\mathbf{w}} < +\infty \text{ a.e. in } \Sigma^3. \quad (6.14)$$

Before we begin the proof of this theorem we wish to establish some short propositions. As in [7], we follow [11] and introduce the cut-off functions

$$u^+ := \begin{cases} 0 & , u \leq 0 \\ u & , u > 0 \end{cases} \quad (6.15)$$

$$u^- := \begin{cases} 0 & , u \geq 0 \\ u & , u < 0 \end{cases} \quad (6.16)$$

for  $u$  in some appropriate function space which makes sense. In this context, for say  $u \in W^{1,2}(\Omega)$ .

**Proposition 6.2.2.**  $(u + v)^+ \leq u^+ + v^+$  and  $(u + v)^- \leq u^- + v^-$

*Proof.* Case 1: Suppose  $u + v \leq 0$ . Without loss, we can assume  $u < -v < 0$ . So in particular,  $u^+ = 0$  and  $v^+ = v$ . Then,

$$0 \leq 0 = (u + v)^+ \leq 0 + v = u^+ + v^+.$$

Case 2: Suppose  $u + v > 0$ . Without loss, we can again assume  $u > -v > 0$ . So that we have  $u^+ = u$  and  $v^+ = 0$ . Then,

$$0 \leq (u + v)^+ = u + v \leq u = u + 0 = u^+ + v^+.$$

The proof of the other case is exactly the same. □

**Proposition 6.2.3.**  $\text{tr}_D u^+ = (\text{tr}_D u)^+$  and  $\text{tr}_D u^- = (\text{tr}_D u)^-$

*Proof.* Case 1: Suppose  $u > 0$ . Then  $\text{tr}_D u > 0$  as well. In which case we have,

$$\text{tr}_D u^+ = \text{tr}_D u = (\text{tr}_D u)^+.$$

Case 2: Suppose  $u \leq 0$ . Likewise,  $\text{tr}_D u \leq 0$  and

$$\text{tr}_D u^+ = \text{tr}_D 0 = 0 = (\text{tr}_D u)^+.$$

Again, the proof of the other case is similar.  $\square$

**Proposition 6.2.4.** *Let  $\phi^+$  and  $\phi^-$  be as in Theorem 6.2.1. Then*

$$(\hat{\nabla}\phi, \hat{\nabla}\phi^+) = \|\hat{\nabla}\phi^+\|_{0,2}^2 \quad (6.17)$$

and

$$(\hat{\nabla}\phi, \hat{\nabla}\phi^-) = \|\hat{\nabla}\phi^-\|_{0,2}^2. \quad (6.18)$$

*Proof.* We just show the first claim; again, the other follows similarly. Put  $A = \{x : \phi(x) > \beta_{\mathbf{w}}\}$ . Then

$$\begin{aligned} \|\hat{\nabla}\phi^+\|_{0,2}^2 &= \int_{\Omega} \hat{\nabla}\phi^+ \cdot \hat{\nabla}\phi^+ \, dx \\ &= \int_A \hat{\nabla}\phi^+ \cdot \hat{\nabla}\phi^+ \, dx + \int_{\Omega \setminus A} \hat{\nabla}\phi^+ \cdot \hat{\nabla}\phi^+ \, dx \\ &= \int_A \hat{\nabla}\phi^+ \cdot \hat{\nabla}\phi^+ \, dx + 0 \\ &= \int_{\Omega} \hat{\nabla}(\phi - \beta_{\mathbf{w}}) \cdot \hat{\nabla}\phi^+ \, dx \\ &= \int_{\Omega} \hat{\nabla}\phi \cdot \hat{\nabla}\phi^+ \, dx. \end{aligned} \quad (6.19)$$

$\square$

The proof of the main theorem, which we are about to present, employs the use of “cut-off” arguments. Generally, such methods are well suited for non-linear problems, e.g. [11].

*Proof.* (Of Theorem 6.2.1)

We begin by establishing the upper-bound using the cut-off function  $u^+$ . Note that  $\phi^+ \in W_D^{1,2}(\Sigma^3)$  since we may write  $\phi = \phi_0 + \phi_1 \in \phi_0 + W_D^{1,2}(\Sigma^3)$  and then

$$0 \leq \phi^+ = (\phi_0 + \phi_1 - \beta_{\mathbf{w}})^+ \leq (\phi_0 - \beta_{\mathbf{w}})^+ + \phi_1^+$$

by Proposition 6.2.2 and

$$\begin{aligned} 0 \leq \text{tr}\phi^+ &\leq \text{tr}_D(\phi_0 - \beta_{\mathbf{w}})^+ + \text{tr}_D\phi_1^+ \\ &\leq (\text{tr}_D(\phi_0 - \beta_{\mathbf{w}}))^+ + (\text{tr}_D\phi_1)^+ = 0 \end{aligned} \quad (6.20)$$

by Proposition 6.2.3 since  $\phi_0 \leq \beta_{\mathbf{w}}$  and  $\phi_1$  is zero on the Dirichlet boundary by choice.

Since  $\phi^+$  is a valid test function, we can plug it into our weak form:

$$(\hat{\nabla}\phi, \hat{\nabla}\phi^+) + (f(\phi, \mathbf{w}), \phi^+) + (K\text{tr}_N\phi - \hat{\phi}_N, \text{tr}_N\phi^+)_N = 0. \quad (6.21)$$

Restricting ourselves to the set  $\{\phi > \beta_{\mathbf{w}}\}$ , i.e. where  $\phi^+$  is non-zero, we may write our weak form as

$$\|\hat{\nabla}\phi^+\|_{0,2}^2 + (f(\phi, \mathbf{w}), \phi^+)_{\{\phi > \beta_{\mathbf{w}}\}} + (K\text{tr}_N\phi^+ - \hat{\phi}_N, \text{tr}_N\phi^+)_N = 0 \quad (6.22)$$

where we have used Proposition 6.2.4 to introduce the norm.

If  $\phi > \beta_{\mathbf{w}}$  and  $a_R \geq 0$ , then

$$f(\phi, \mathbf{w}) \geq h_+(\phi, \mathbf{w}) \quad (\text{by construction}) \quad (6.23)$$

$$\geq h_+(\beta_{\mathbf{w}}, \mathbf{w}) \quad (\text{by uniqueness of root}) \quad (6.24)$$

$$= 0. \quad (6.25)$$

Likewise when  $\phi > \beta_{\mathbf{w}}$  and  $a_R < 0$ , we have

$$f(\phi, \mathbf{w}) \geq h_-(\phi, \mathbf{w}) \geq h_-(\beta_{\mathbf{w}}, \mathbf{w}) = 0. \quad (6.26)$$

In any case, since  $\phi^+ \geq 0$ ,

$$(f(\phi, \mathbf{w}), \phi^+)_{\{\phi > \beta_{\mathbf{w}}\}} \geq 0. \quad (6.27)$$

With this, along with the assumption on the Robin boundary data, we can conclude

$$\|\hat{\nabla} \phi^+\|_{0,2}^2 \leq 0. \quad (6.28)$$

Finally, since we assumed that  $\partial\Omega \neq \emptyset$  (§1) the Poincaré inequality (§7 Appendix) gives us

$$0 \geq \|\hat{\nabla} \phi^+\|_{0,2}^2 \geq C_P \|\phi^+\|_{0,2}^2 \geq 0, \quad (6.29)$$

where  $C_P$  is just a positive constant. It follows that  $\phi^+ = 0$ , since norms are non-degenerate. Thus,  $\phi \leq \beta_{\mathbf{w}}$ .

A similar argument establishes  $\alpha_{\mathbf{w}} \leq \phi$  by using the cut-off function  $u^-$ . We put  $\phi^- = (\phi - \alpha_{\mathbf{w}})^-$  and by Propositions 6.2.2 and 6.2.3 we have  $\phi^- \in W_D^{1,2}(\Sigma^3)$ . By restricting to the set  $\{\phi < \alpha_{\mathbf{w}}\}$ ,

$$\|\hat{\nabla} \phi^-\|_{0,2}^2 + (f(\phi, \mathbf{w}), \phi^-)_{\{\phi < \alpha_{\mathbf{w}}\}} + (K \text{tr}_N \phi^- - \hat{\phi}_N, \text{tr}_N \phi^-)_N = 0. \quad (6.30)$$

Similarly, we conclude

$$f(\phi, \mathbf{w}) \leq g_{\pm}(\phi, \mathbf{w}) \leq g_{\pm}(\alpha_{\mathbf{w}}, \mathbf{w}) = 0. \quad (6.31)$$

Then since  $\phi^-$  is negative on the set  $\{\phi < \alpha_{\mathbf{w}}\}$  the result follows because

$$(f(\phi, \mathbf{w}), \phi^-)_{\{\phi < \alpha_{\mathbf{w}}\}} \geq 0. \quad (6.32)$$

□

## 6.2.1 A Priori Global $\mathbf{w}$ -Independent Lower $L^\infty$ -Estimates

In addition to these *local*  $\mathbf{w}$ -dependent roots, we can find *global* lower bounds if we impose the condition  $\inf a_\rho > 0$  when we are in the case of  $a_R \geq 0$ .

**Corollary 6.2.1.** *There exists a global  $\alpha > 0$  such that  $\alpha < \alpha_{\mathbf{w}}$  as long as  $a_R < 0$  or if  $a_R \geq 0$ , then  $\inf a_\rho > 0$ .*

*Proof.* Suppose  $a_R < 0$ . Then consider the function  $\tilde{g}(\phi) = \sup a_\tau \phi^{12} - \inf |a_R| \phi^8$ . Now since  $a_R$  is strictly negative  $\inf |a_R| \neq 0$  and,

$$\begin{aligned} g_-(\phi, \mathbf{w}) &= \sup a_\tau \phi^{12} - \inf |a_R| \phi^8 - \inf a_\rho \phi^4 - \inf a_{\mathbf{w}}, \\ &\leq \sup a_\tau \phi^{12} - \inf |a_R| \phi^8, \\ &=: \tilde{g}(\phi). \end{aligned} \quad (6.33)$$



Thus by Lemma 6.2.1 and the Intermediate Value Theorem, there exists  $\alpha > 0$  such that  $\tilde{g}(\alpha) = 0$ . In particular,  $g_-(\alpha, \mathbf{w}) \leq 0$  and so  $0 < \alpha \leq \alpha_{\mathbf{w}} \leq \phi$ .

For the other case, re-define  $\tilde{g}(\phi) = \sup a_{\tau} \phi^{12} + \sup a_R \phi^8 - \inf a_{\rho} \phi^4$ . Note that the assumption on  $a_{\rho}$  ensures that  $\tilde{g}$  has a root. The proof is then similar to that above.  $\square$

### 6.3 Existence of Weak Solutions

Now we move on to establishing the existence of solutions to the Hamiltonian constraint. Using the calculus of variations, we will show that (weak) solutions exist, although they may not be, in general, unique. However, when  $a_R \geq 0$ , uniqueness is not hard to show; cf. Theorem 6.3.3. We use the convex analysis framework in [3] to prove our main result in this section, Theorem 6.3.2.

#### Weak formulation arising as an Euler condition

Due to the fact that the principle parts of the Hamiltonian constraint operators produced by the conformal decomposition are formally self-adjoint, the weak formulation arises naturally as the Euler condition for stationarity of an associated (energy) functional. See [7] for a discussion of the variational principles for the constraints. It is easy to verify

$$J_{\mathbb{H}}(\phi) := \frac{1}{2}(\hat{\nabla}\phi, \hat{\nabla}\phi) + (G(\phi), 1) + \frac{1}{2}(K\text{tr}_N\phi - \hat{\phi}_N, \text{tr}_N\phi)_N =: \int_{\Omega} \mathfrak{L}(\hat{\nabla}\phi, \phi, x) dx \quad (6.34)$$

gives rise to the weak form (6.11), where  $G(\phi) := \frac{1}{2}a_R\phi^2 + \frac{1}{6}a_{\tau}\phi^6 + \frac{1}{2}a_{\rho}\phi^{-2} + \frac{1}{6}a_{\mathbf{w}}\phi^{-6}$  (for which the Gâteaux derivative is  $f(\phi, \mathbf{w})$ ). That is to say, computing the stationary points of the first variation

$$\delta J_{\mathbb{H}}(\phi) = \frac{d}{d\epsilon} J_{\mathbb{H}}(\phi + \epsilon\psi)|_{\epsilon=0} = 0 \quad (6.35)$$

is equivalent to finding a solution to the weak form

$$\mathbb{H}(\phi)(\psi) = 0$$

as long as we can show that the energy is well-defined, i.e. finite, on

$$\mathcal{B} := \{\phi \in \phi_0 + W_D^{1,2}(\Omega) : 0 < \alpha_{\mathbf{w}} \leq \phi \leq \beta_{\mathbf{w}} < +\infty \text{ a.e. in } \Omega\}. \quad (6.36)$$

#### Weak form and energy functional are well-defined

The following two results are similar to those found in [7].

**Lemma 6.3.1.** *Let  $f(\phi, \mathbf{w})$  be as in (4.19), and  $G(\phi)$  as above. Then for all  $\phi \in \mathcal{B}$  and  $\psi \in W_D^{1,2}(\Omega)$ ,*

$$f(\phi, \mathbf{w}) \in [W^{1,2}(\Omega)]^*$$

and

$$G(\phi) \in L^1(\Omega).$$

*Proof.* Let us recall that we have the source terms

$$\tau \in L^{12/5}(\Omega), \quad \hat{\rho} \in L^{6/5}(\Omega), \quad \hat{\sigma}_*^{ij} \in L^{12/5}(\Omega), \quad \mathbf{w} \in W^{1,12/5}(\Omega).$$

In particular, we have

$$\|a_{\tau}\|_{0,6/5}, \quad \|a_{\rho}\|_{0,6/5}, \quad \|a_{\mathbf{w}}\|_{0,6/5}$$

all being finite. Now with repeated use of Hölder inequalities, we have

$$\begin{aligned}
\int_{\Omega} f(\phi, \mathbf{w})\psi \, dx &= \int_{\Omega} (a_R\phi + a_{\tau}\phi^5 - a_{\rho}\phi^{-3} - a_{\mathbf{w}}\phi^{-7})\psi \, dx \\
&\leq \|a_R\phi\psi\|_{0,1} + \|a_{\tau}\phi^5\psi\|_{0,1} + \|a_{\rho}\phi^{-3}\psi\|_{0,1} + \|a_{\mathbf{w}}\phi^{-7}\psi\|_{0,1} \\
&\leq \beta_{\mathbf{w}}\|a_R\psi\|_{0,1} + \beta_{\mathbf{w}}^5\|a_{\tau}\psi\|_{0,1} + \alpha_{\mathbf{w}}^{-3}\|a_{\rho}\psi\|_{0,1} + \alpha_{\mathbf{w}}^{-7}\|a_{\mathbf{w}}\psi\|_{0,1} \\
&\leq (\beta_{\mathbf{w}}\|a_R\|_{0,6/5} + \beta_{\mathbf{w}}^5\|a_{\tau}\|_{0,6/5} + \alpha_{\mathbf{w}}^{-3}\|a_{\rho}\|_{0,6/5} + \alpha_{\mathbf{w}}^{-7}\|a_{\mathbf{w}}\|_{0,6/5})\|\psi\|_{0,6}.
\end{aligned}$$

Since we have the embedding of  $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ , there exists some constant  $C_6$  such that  $\|\cdot\|_{0,6} \leq C_6\|\cdot\|_{1,2}$ . So we take

$$C := C_6(\beta_{\mathbf{w}}\|a_R\|_{0,6/5} + \beta_{\mathbf{w}}^5\|a_{\tau}\|_{0,6/5} + \alpha_{\mathbf{w}}^{-3}\|a_{\rho}\|_{0,6/5} + \alpha_{\mathbf{w}}^{-7}\|a_{\mathbf{w}}\|_{0,6/5}) \quad (6.37)$$

and conclude that  $f(\phi, \mathbf{w}) \in [W^{1,2}(\Omega)]^*$ .

To show the other claim, note that

$$\begin{aligned}
\int_{\Omega} G(\phi) \, dx &= \int_{\Omega} \left( \frac{a_R}{2}\phi^2 + \frac{a_{\tau}}{6}\phi^6 + \frac{a_{\rho}}{2}\phi^{-2} + \frac{a_{\mathbf{w}}}{6}\phi^{-6} \right) dx \\
&\leq \frac{1}{2}\|a_R\|_{0,1}\beta_{\mathbf{w}}^2 + \frac{1}{6}\|a_{\tau}\|_{0,1}\beta_{\mathbf{w}}^6 + \frac{1}{2}\|a_{\rho}\|_{0,1}\alpha_{\mathbf{w}}^{-3} + \frac{1}{6}\|a_{\mathbf{w}}\|_{0,1}\alpha_{\mathbf{w}}^{-6}.
\end{aligned}$$

Then we use the embedding  $L^{6/5}(\Omega) \hookrightarrow L^1(\Omega)$  to find the constant  $C_1$  such that  $\|\cdot\|_{0,1} \leq C_1\|\cdot\|_{0,6/5}$ . Hence it follows that  $G(\phi) \in L^1(\Omega)$ .  $\square$

N.B. Just two constants  $c_1, c_2$  with  $0 < c_1 \leq \phi \leq c_2$  were needed to complete the proof; there was nothing special about  $\alpha_{\mathbf{w}}$  and  $\beta_{\mathbf{w}}$ .

**Theorem 6.3.1.** *The Hamiltonian constraint in weak form  $\mathbb{H}(\phi)(\psi)$  is well-defined on  $\mathcal{B} \times W_D^{1,2}(\Omega)$  and the associated energy functional  $J_{\mathbb{H}}$  is well-defined on  $\mathcal{B}$ .*

*Proof.* The result follows almost immediately from Lemma 6.3.1. Begin by recalling

$$\begin{aligned}
\mathbb{H}(\phi)(\psi) &= a_L(\phi, \psi) + F_f(\phi, \mathbf{w})(\psi) \\
&= (\hat{\nabla}\phi, \hat{\nabla}\psi) + (K\mathbf{tr}_N\phi, \mathbf{tr}_N\psi)_N + (f(\phi, \mathbf{w}), \psi) - (\hat{\phi}_N, \mathbf{tr}_N\psi)_N \\
&= (\hat{\nabla}\phi, \hat{\nabla}\psi) + (K\mathbf{tr}_N\phi - \hat{\phi}_N, \mathbf{tr}_N\psi)_N + (f(\phi, \mathbf{w}), \psi).
\end{aligned} \quad (6.38)$$

Then with the application of Hölder inequalities to each component on the right:

$$(\hat{\nabla}\phi, \hat{\nabla}\psi)^2 \leq |\phi|_{1,2}^2|\psi|_{1,2}^2 \leq (|\phi|_{1,2}^2 + \|\phi\|_{0,2}^2)(|\psi|_{1,2}^2 + \|\psi\|_{0,2}^2) := \|\phi\|_{1,2}^2\|\psi\|_{1,2}^2. \quad (6.39)$$

where we use a single bar, as opposed to double bars, to denote the  $W^{1,2}(\Omega)$  semi-norm. Similarly,

$$\begin{aligned}
(K\mathbf{tr}_N\phi - \hat{\phi}_N, \mathbf{tr}_N\psi)_N &\leq \|K\|_{0,\infty}\|\mathbf{tr}_N\phi - \hat{\phi}_N\|_{0,2}\|\mathbf{tr}_N\psi\|_{0,2} \\
&\leq C_{1/2}^2\|K\|_{0,\infty}\|\mathbf{tr}_N\phi - \hat{\phi}_N\|_{1/2,2}\|\mathbf{tr}_N\psi\|_{1/2,2} \\
&\leq 2C_{1/2}^2C_{\text{tr}}^2\|K\|_{0,\infty}\|\phi\|_{1,2}\|\hat{\phi}_N\|_{1,2}\|\psi\|_{1,2}
\end{aligned} \quad (6.40)$$

where in the second line we made use of the embedding  $W^{1/2,2}(\Omega) \hookrightarrow L^2(\Omega)$  to find a constant  $C_{1/2}$  such that  $\|\cdot\|_{0,2} \leq C_{1/2}\|\cdot\|_{1/2,2}$  and in the third line, the definition of the trace function which embeds  $W^{1,2}(\Omega) \hookrightarrow W^{1/2,2}(\partial\Omega)$  to find the constant  $C_{\text{tr}}$ . Finally, by Lemma 6.3.1,  $f(\phi, \mathbf{w}) \in [W^{1,2}(\Omega)]^*$ . It then follows

$$\mathbb{H}(\phi)(\psi) \leq C\|\phi\|_{1,2}\|\psi\|_{1,2}. \quad (6.41)$$

Thus  $\mathbb{H}(\phi)(\psi)$  is finite on  $\mathcal{B} \times W_D^{1,2}(\Omega)$ .

The proof that  $J_{\mathbb{H}}$  is finite is nearly identical.  $\square$

### Existence of Weak Solutions Via Convex Analysis

At this point we wish to take a step back and think of our energy functional  $J_{\mathbb{H}}$  more abstractly. More precisely, just as in §3.2, in which we discussed the Lagrangian formulation of the Einstein equations, we will view the energy functional as an action of a Lagrangian density. Hence the definition

$$J_{\mathbb{H}}(\phi) := \int_{\Omega} \mathfrak{L}(\hat{\nabla}\phi, \phi, x) \, dx \quad (6.42)$$

as in (6.34).

**Definition 6.3.1.** For all  $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{B}$  such that

$$\phi_n \rightharpoonup \phi \implies J_{\mathbb{H}}(\phi) \leq \liminf_{n \rightarrow \infty} J_{\mathbb{H}}(\phi_n) \quad \forall \phi \in \mathcal{B} \quad (6.43)$$

we say  $\{\phi_n\}$  is (sequentially) weak lower-semicontinuous.

**Theorem 6.3.2.** If  $J_{\mathbb{H}} : \mathcal{B} \rightarrow \mathbb{R}$  is a weak lower-semicontinuous functional such that

$$J_{\mathbb{H}}(\phi) \geq \gamma \int_{\Omega} |\hat{\nabla}\phi|^2 \, dx - \lambda|\Omega| \quad (6.44)$$

for  $\gamma > 0$ ,  $\lambda \geq 0$  and the mapping

$$\hat{\nabla}\phi \mapsto \mathfrak{L}(\hat{\nabla}\phi, \phi, x) \quad (6.45)$$

is convex on  $\mathcal{B}$ , then  $J_{\mathbb{H}}(\cdot)$  has a local minimizer  $\phi^* \in \mathcal{B}$ , i.e.  $J_{\mathbb{H}}(\phi^*) \leq J_{\mathbb{H}}(\nu)$  for all  $\nu \in \mathcal{B}$ .

*Proof.* The proof is a variation of one found in [3]. Without loss we can assume  $\lambda = 0$ , for else consider  $\mathfrak{L} + \lambda$ . Begin by setting  $m = \inf_{\nu \in \mathcal{B}} J_{\mathbb{H}}(\nu)$ . If  $m = +\infty$  we are done, so we may as well as assume that  $m$  is finite. By definition, we can pick a sequence  $\{\phi_n^*\}_{n=1}^{\infty} \subset \mathcal{B}$  such that

$$J_{\mathbb{H}}(\phi_n^*) \rightarrow m. \quad (6.46)$$

Since we have

$$J_{\mathbb{H}}(\phi_n^*) \geq \gamma \int_{\mathcal{M}} |\hat{\nabla}\phi_n^*|^2 \, dx \quad (6.47)$$

and  $m$  is finite, it follows that

$$\sup_n \|\hat{\nabla}\phi_n^*\|_{0,2} < +\infty. \quad (6.48)$$

Fixing any function  $\nu \in \mathcal{B}$  yields  $\phi_n^* - \nu \in \mathcal{B}$ . Then using the triangle inequality and Poincaré inequality we have

$$\begin{aligned} \|\phi_n^*\|_{0,2} &\leq \|\phi_n^* - \nu\|_{0,2} + \|\nu\|_{0,2} \\ &\leq C_P \|\hat{\nabla}(\phi_n^* - \nu)\|_{0,2} + \|\nu\|_{0,2} \\ &\leq C_P \|\hat{\nabla}(\phi_n^* - \nu)\|_{0,2} + C_2 \|\nu\|_{1,2} < +\infty \end{aligned} \quad (6.49)$$

where in the last line we use (6.48) the embedding  $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$  to find the constant  $C_2$  such that  $\|\cdot\|_{0,2} \leq C_2 \|\cdot\|_{1,2}$ . Hence we can conclude that

$$\sup_n \|\phi_n^*\|_{0,2} < +\infty. \quad (6.50)$$

Along with (6.48), we have  $\{\phi_n^*\}_{n=1}^\infty$  bounded in  $W^{1,2}(\Omega)$ . Now we can extract a subsequence  $\{\phi_{n_j}^*\}_{j=1}^\infty \subset \{\phi_n^*\}_{n=1}^\infty$  and a function  $\phi^* \in W^{1,2}(\Omega)$  such that

$$\phi_{n_j}^* \rightharpoonup \phi^* \quad (\text{weakly}) \quad \text{in } W_D^{1,2}(\Omega). \quad (6.51)$$

Since  $\mathcal{B}$  is closed, in  $W^{1,2}(\Omega)$ , and convex, Mazur's Theorem (§7 Appendix) implies  $\mathcal{B}$  is weakly closed. Thus  $\phi^* - \nu \in \mathcal{B}$  and so  $\phi^* \in \mathcal{B}$ . By assumption (weak lower-semicontinuity) we know that  $J_{\mathbb{H}}(\phi^*) \leq m$  and since  $\phi^* \in \mathcal{B}$  we have

$$J_{\mathbb{H}}(\phi^*) = m = \inf_{\nu \in \mathcal{B}} J_{\mathbb{H}}(\nu). \quad (6.52)$$

□

The mapping defined in Theorem 6.3.2 is indeed convex in the variable  $\hat{\nabla}\phi$  since it appears quadratically in  $J_{\mathbb{H}}$ . However, the one assumption which is not obvious is the weak lower-semicontinuity. In fact, this usually is one of the most difficult properties to establish in convex analysis. Most known results require the energy functional  $J_{\mathbb{H}}$  to be convex in each variable, e.g. in [20]. We, indeed, have this if we assume  $a_R \geq 0$ . This was the approach used in [7], restricting the result there to the convex case. But if  $a_R$  becomes too negative,  $J_{\mathbb{H}}$  will fail to remain convex. In the result which follows, adapted from [3], we show that this global convexity condition is sufficient but not necessary. As it turns out, convexity in the gradient variable is enough to establish weak lower-semicontinuity as long as the Lagrangian density is bounded from below. The latter is immediate since  $J_{\mathbb{H}}$  is defined only on  $\mathcal{B}$ .

**Lemma 6.3.2.** *Suppose  $\mathfrak{L}(\hat{\nabla}\phi, \phi, x)$  is bounded below and the mapping*

$$\hat{\nabla}\phi \mapsto \mathfrak{L}(\hat{\nabla}\phi, \phi, x) \quad (6.53)$$

*is convex for each  $\phi$  and  $x$ . Then  $J_{\mathbb{H}}(\cdot)$  is weak lower-semicontinuous on  $W^{1,2}(\Omega)$ .*

*Proof.* Choose a weakly convergent sequence such that  $\phi_n \rightharpoonup \phi$  in  $W^{1,2}(\Omega)$ . Put  $m := \liminf_{n \rightarrow \infty} J_{\mathbb{H}}(\phi_n)$ . Since every weakly convergent sequence is bounded,

$$\sup_n \|\phi_n\|_{1,2} < \infty. \quad (6.54)$$

Thus, we can assume (passing to a subsequence if necessary) that

$$m = \lim_{n \rightarrow \infty} J_{\mathbb{H}}(\phi_n). \quad (6.55)$$

Since the embedding  $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$  is compact (again, passing to a subsequence if necessary)

$$\phi_n \rightarrow \phi \quad (\text{strongly}) \quad \text{a.e. on } \Omega. \quad (6.56)$$

Let  $\epsilon > 0$  be given. According to Egoroff's Theorem (§7 Appendix) we can find a measurable set  $\Omega_\epsilon$  with  $\mu(\Omega_\epsilon) \leq \epsilon$  such that

$$\phi_n \rightarrow \phi \quad (\text{uniformly}) \quad \text{in } \Omega_\epsilon^c. \quad (6.57)$$

Now put

$$\mathcal{N}_\epsilon := \left\{ \phi \in W^{1,2}(\Omega) : |\phi| + |\hat{\nabla}\phi| \leq \frac{1}{\epsilon} \right\}. \quad (6.58)$$

Note  $\mu(\Omega \setminus \mathcal{N}_\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Similarly when we put

$$\mathcal{G}_\epsilon := \Omega_\epsilon^c \cap \mathcal{N}_\epsilon \quad (6.59)$$

we have  $\mu(\Omega \setminus \mathcal{G}_\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

Since  $\mathfrak{L}$  is bounded, we may assume without loss that  $\mathfrak{L} \geq 0$ . Then

$$\begin{aligned} J_{\mathbb{H}}(\phi_n) &= \int_{\Omega} \mathfrak{L}(\hat{\nabla}\phi_n, \phi_n, x) \, dx \\ &\geq \int_{\mathcal{G}_\epsilon} \mathfrak{L}(\hat{\nabla}\phi_n, \phi_n, x) \, dx \\ &\geq \int_{\mathcal{G}_\epsilon} \mathfrak{L}(\hat{\nabla}\phi, \phi_n, x) \, dx + \int_{\mathcal{G}_\epsilon} \partial_{\hat{\nabla}\phi} \mathfrak{L}(\hat{\nabla}\phi, \phi_n, x) \cdot (\hat{\nabla}\phi_n - \hat{\nabla}\phi) \, dx \end{aligned}$$

where the last line follows from the convexity condition (see §7 Appendix). Appealing to the Dominated Convergence Theorem (§7), and with the use of (6.57) we have

$$\lim_{n \rightarrow \infty} \int_{\mathcal{G}_\epsilon} \mathfrak{L}(\hat{\nabla}\phi, \phi_n, x) \, dx = \int_{\mathcal{G}_\epsilon} \mathfrak{L}(\hat{\nabla}\phi, \phi, x) \, dx. \quad (6.60)$$

Since  $\partial_{\hat{\nabla}\phi} \mathfrak{L}(\hat{\nabla}\phi, \phi_n, x) \rightarrow \partial_{\hat{\nabla}\phi} \mathfrak{L}(\hat{\nabla}\phi, \phi, x)$  uniformly on  $\mathcal{G}_\epsilon$ , and  $\hat{\nabla}\phi_n \rightharpoonup \hat{\nabla}\phi$  weakly in  $L^2(\Omega)$  we have

$$\lim_{n \rightarrow \infty} \int_{\mathcal{G}_\epsilon} \partial_{\hat{\nabla}\phi} \mathfrak{L}(\hat{\nabla}\phi, \phi_n, x) \cdot (\hat{\nabla}\phi_n - \hat{\nabla}\phi) \, dx = 0. \quad (6.61)$$

Thus we conclude

$$m = \lim_{n \rightarrow \infty} J_{\mathbb{H}}(\phi_n) \geq \int_{\mathcal{G}_\epsilon} \mathfrak{L}(\hat{\nabla}\phi, \phi, x) \, dx. \quad (6.62)$$

Now letting  $\epsilon$  tend to zero, an application of the Monotone Convergence Theorem (§7) yields

$$m \geq \int_{\Omega} \mathfrak{L}(\hat{\nabla}\phi, \phi, x) \, dx = J_{\mathbb{H}}(\phi) \quad (6.63)$$

which is what we sought to prove.  $\square$

### Uniqueness of weak solution

From these very general results we cannot, in general, guarantee uniqueness of our weak solution  $\phi$ . However, if it is the case that  $a_R \geq 0$ , uniqueness does follow since the entire functional  $J_{\mathbb{H}}(\phi)$  is convex and  $f(\phi, \mathbf{w})$  is monotonically non-negative [7].

**Theorem 6.3.3.** *If in addition to the assumptions made in Theorem 6.3.2, we assume  $a_R \geq 0$ , then the minimizing solution is unique.*

*Proof.* Suppose there were two solutions, say  $\phi_1$  and  $\phi_2$ , then we have

$$(\mathbb{H}(\phi_1) - \mathbb{H}(\phi_2))(\psi) = 0, \quad \forall \psi \in W_D^{1,2}(\Omega). \quad (6.64)$$

Then by taking  $\psi := \phi_1 - \phi_2 \in W_D^{1,2}(\Omega)$  we are left with,

$$\begin{aligned} 0 &= (\mathbb{H}(\phi_1) - \mathbb{H}(\phi_2))(\phi_1 - \phi_2) \\ &= (\hat{\nabla}(\phi_1 - \phi_2), \hat{\nabla}(\phi_1 - \phi_2)) + F_f(\phi_1, \mathbf{w})(\phi_1 - \phi_2) - F_f(\phi_2, \mathbf{w})(\phi_1 - \phi_2) \\ &\geq 0 \end{aligned} \quad (6.65)$$

where in the last line we used the assumptions on the Robin data and the fact that  $\partial_\phi f(\phi, \mathbf{w}) = a_R + 5a_\tau\phi^4 + 3a_\rho\phi^{-4} + 7a_{\mathbf{w}}\phi^{-8} \geq 0$ . We conclude using the Poincaré inequality that

$$0 = \|\phi_1 - \phi_2\|_{1,2}^2 = \frac{1}{2}\|\phi_1 - \phi_2\|_{1,2}^2 + \frac{1}{2}\|\phi_1 - \phi_2\|_{1,2}^2 \geq \min\left\{\frac{1}{2}, \frac{C_P}{2}\right\} \|\phi_1 - \phi_2\|_{1,2}^2 \geq 0. \quad (6.66)$$

Since norms are non-degenerate, we have  $\phi_1 = \phi_2$ . □

# Chapter 7

## Appendix

To see discussions and/or proofs on the following, refer to any standard textbook on the subject, e.g. see [3, 4, 17, 19].

### 7.1 Functional Analysis

**Theorem 7.1.1** (Mazur's Theorem). *Let  $C$  be a convex subset of a Banach space  $X$ . Then the norm closure  $\overline{C}$  equals the weak closure  $\overline{C}^w$ .*

**Theorem 7.1.2** (Supporting Hypersurfaces). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function, i.e.*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (7.1)$$

*for all  $x, y \in \mathbb{R}^n$  and  $0 \leq t \leq 1$ . Then for each  $x \in \mathbb{R}^n$  there exists  $D \in \mathbb{R}^n$  such that*

$$f(y) - f(x) \geq D \cdot (y - x) \quad (7.2)$$

*for all  $y \in \mathbb{R}^n$ . If  $f$  is differentiable, then  $D = \nabla f$ .*

### 7.2 Measure Theory

**Theorem 7.2.1** (Monotone Convergence Theorem). *If  $f_n$  is a sequence of positive real measurable functions such that  $f_j \leq f_{j+1}$  for all  $j$ , and  $f = \lim_{n \rightarrow \infty} f_n$ , then*

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

**Theorem 7.2.2** (Dominated Convergence Theorem). *If  $f_n$  is a sequence of  $L^1$  functions such that  $f_n \rightarrow f$  a.e. and there exists a non-negative  $g \in L^1$  such that  $|f_n| \leq g$  a.e. for all  $n$ , then  $f \in L^1$  and*

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

**Theorem 7.2.3** (Egoroff's Theorem). *Let  $(X, \mathcal{M}, \mu)$  be a finite measure space, that is,  $\mu(X) < \infty$ . Suppose  $f_n$  is a sequence of measurable complex-valued functions on  $X$  such that  $f_n \rightarrow f$  a.e. Then for every  $\epsilon > 0$  there exists  $E \subset X$  measurable such that  $\mu(E) < \epsilon$  and  $f_n \rightarrow f$  uniformly on  $E^c$ .*

## 7.3 Partial Differential Equation Theory

**Definition 7.3.1** (Poincaré Inequality). Let  $\Omega \subset \mathbb{R}$  be a bounded set. For all  $u \in W_0^{1,p}(\Omega)$ ,  $1 \leq p < \infty$ , there exists a constant  $C := C(\Omega, p)$  such that

$$\|u\|_{0,p}^p \leq C \|Du\|_{0,p}^p. \quad (7.3)$$

## 7.4 Differential Geometry

**Definition 7.4.1** (Exterior Derivative). Let  $\bigwedge \mathcal{M} := \bigoplus_{p=0}^n \bigwedge^p \mathcal{M}$  denote the exterior algebra of differential forms on  $\mathcal{M}$ . Then there exists a unique operator

$$d : \bigwedge \mathcal{M} \rightarrow \bigwedge \mathcal{M} \quad (7.4)$$

such that

- i.)  $d$  is an anti-derivation of degree 1 on  $\bigwedge \mathcal{M}$
- ii.)  $d \circ d = 0$
- iii.)  $\langle df, X \rangle = Xf$  for all  $f \in C^\infty(\mathcal{M})$  and all  $X \in \mathfrak{X}(\mathcal{M})$ .

$d$  is then called the **exterior derivative**.

**Definition 7.4.2** (Absolute Exterior Differential). For every tensor valued  $p$ -form  $T$  of type  $(r, s)$ , there is a unique tensor valued  $(p+1)$ -form  $DT$  of type  $(r, s)$  such that

$$(DT)^{i_1 \dots i_r}_{j_1 \dots j_s} = dT^{i_1 \dots i_r}_{j_1 \dots j_s} + \sum_{k=1}^r \omega^k_b \wedge T^{b i_1 \dots \widehat{i}_k \dots i_r}_{j_1 \dots j_s} - \sum_{k=1}^s \omega^b_{j_k} \wedge T^{i_1 \dots i_r}_{b j_1 \dots \widehat{j}_k \dots j_s}. \quad (7.5)$$

$DT$  is called the **absolute exterior differential of  $T$** .

**Definition 7.4.3** (Interior Product). Let  $A$  be a commutative, associative, unitary  $\mathbb{R}$ -algebra and  $E$  be an  $A$ -module. Then for each  $p \in \mathbb{N}$  the mapping

$$E \times \bigwedge^p E \rightarrow \bigwedge^{p-1} E$$

such that  $(v, \omega) \mapsto \iota_v \omega$  where

$$\begin{aligned} (\iota_v \omega)(v_1, \dots, v_{p-1}) &= \omega(v, v_1, \dots, v_{p-1}) \\ \iota_v \omega &= 0, \quad \forall \omega \in \bigwedge^0 E, \end{aligned}$$

$\iota_v \omega$  is called the **inner product of  $v$  and  $\omega$** .

**Definition 7.4.4** (Lie Derivative). For  $\omega \in \bigwedge^p(\mathcal{M})$ ,  $X \in \mathfrak{X}(\mathcal{M})$  let  $\iota_X \omega$  denote the interior product of  $X$  and  $\omega$ . Then the **Lie derivative of  $\omega$  with respect to  $X$**  is defined as

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega. \quad (7.6)$$

For a tensor  $T$  of type  $(r, s)$ , we have

$$\mathcal{L}_X T^{i_1 \dots i_r}_{j_1 \dots j_s} = X_k \nabla_k T^{i_1 \dots i_r}_{j_1 \dots j_s} + \sum_{k=1}^r (\nabla_{j_k} X^b) T^{i_1 \dots i_r}_{b j_1 \dots \widehat{j}_k \dots j_s} - \sum_{k=1}^s (\nabla_b X^{i_k}) T^{b i_1 \dots \widehat{i}_k \dots i_r}_{j_1 \dots j_s}. \quad (7.7)$$



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