

ON GENUS ZERO STABLE MAPS TO THE FLAG VARIETY

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ABSTRACT. In this paper we calculate the Euler characteristic of the moduli stack of genus zero stable maps into the Flag Variety through the use of Białynicki-Birula localization for flags of length ℓ without markings in multi-degrees $d_\bullet = (1, 0)$; $(1, 1)$; and $(1, 2)$ for $\ell = 2$ and $d_\bullet = (1, 1, 1)$ for $\ell = 3$. For $d_\bullet = (1, 0)$; and $(1, 1)$ we give a generating series for when the number of marked points is greater than or equal to three.

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1. Introduction. The moduli space of stable maps, first proposed by Kontsevich and Manin, has been an increasingly useful tool in solving enumerative calculations in algebraic geometry. This space arises as a natural compactification of the space of rational curves of degree d in a smooth projective variety.

This paper builds off of key results from a previous paper due to Agrawal [A] which calculated the Euler characteristic of the stack of stable maps to Grassmannians in degrees 1, 2, and 3. More recently, the full Poincaré polynomial of stable maps to Grassmannians has been computed in degrees 1, 2, and 3 by López Martín [L]. Both papers considered curves of genus zero and with zero markings.

In this paper we calculate the Euler characteristic of the moduli stack of stable maps to a generalization of the Grassmannian, the Flag Variety. We work exclusively over \mathbb{C} , the field of complex numbers. The methods employed are based on those of Oprea who showed that the Euler characteristic of the stack of stable maps can be determined from the fixed locus [O].

We first review some basic facts about Grassmannians and Flag Varieties, compute their Euler characteristics, and give a calculation of their dimension in sections 3 and 4. In section 5, we discuss the Plücker embedding of the Grassmannian and show that it can be generalized to the Flag Variety. We then introduce the non-compact space of morphisms in genus zero to the Flag Variety, calculate its Euler characteristic and finally, in section 9 and 10, we introduce its compactification by the space of stable maps and perform calculations of its Euler characteristic for several different curve classes both with and without markings. As an appendix, we have also included a calculation of the Poincaré polynomial of the Flag Variety.

2. Localization. Our method for calculating the Euler characteristic stems from Białynicki-Birula localization. The following theorem can be found in [A] and is a consequence of [O, lemma 6]

Theorem 1. *If \mathcal{X} is a smooth Deligne-Mumford stack which admits a torus action of \mathbb{C}^* with fixed locus $\mathcal{X}^{\mathbb{C}^*}$, then $\chi(\mathcal{X}) = \chi(\mathcal{X}^{\mathbb{C}^*})$*

The localization theorem is most useful when the \mathbb{C}^* -fixed locus of \mathcal{X} is a finite collection of discrete points. In these cases the Euler characteristic counts the number of points in the fixed loci of the torus action. This is because the Euler characteristic is additive for disjoint sets and equal to one for a single point. In these cases the Euler characteristic is given by

$$\chi(\mathcal{X}) = \left| \mathcal{X}^{\mathbb{C}^*} \right|$$

3. The Flag Variety over \mathbb{C}^n . The Flag Variety parametrizes strictly increasing sequences of linear subspaces of \mathbb{C}^n . Given integers $0 = k_0 < k_1 < \dots < k_\ell < k_{\ell+1} = n$, a point in $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$ is given by a *flag*,

$$V_\bullet = \{\emptyset \subset V_1 \subset \dots \subset V_\ell \subset \mathbb{C}^n \mid \dim V_i = k_i\}.$$

We can simplify some expressions by defining the remainders at each step,

$$r_i = k_i - k_{i-1}, \text{ for } 1 \leq i \leq \ell + 1$$

so that for each i ,

$$k_i = \sum_{j=1}^i r_j$$

Example 1. First, we give a calculation of the Euler characteristic of the Flag Variety via the localization theorem. We fix once and for all a basis e_1, \dots, e_n for \mathbb{C}^n . Define a *torus action*, of \mathbb{C}^* on \mathbb{C}^n with weights $\lambda_1, \dots, \lambda_n$ by

$$t \cdot (z_1, \dots, z_n) = (t^{\lambda_1} z_1, \dots, t^{\lambda_n} z_n)$$

This induces a \mathbb{C}^* -action on $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$ in the following manner,

$$t \cdot V_\bullet = \{\emptyset \subset t \cdot V_1 \subset \dots \subset t \cdot V_\ell \subset \mathbb{C}^n\}, \text{ where } t \cdot V_i = \{t \cdot v_i \mid v_i \in V_i\}.$$

A given subspace $V \subset \mathbb{C}^n$ of dimension k is a *fixed point* of the \mathbb{C}^* action if

$$t \cdot V = V, \forall t \in \mathbb{C}^*$$

This occurs if and only if V is spanned by precisely k basis elements. Thus we can index all of the k -dimensional fixed subspaces of \mathbb{C}^n by subsets,

$$I \subset [n] := \{1, 2, \dots, n\} \text{ such that } |I| = k.$$

Thus the fixed points in $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$ can be indexed by *flags of sets*,

$$I_\bullet = \{\emptyset \subset I_1 \subset \dots \subset I_\ell \subset [n] \mid |I_i| = k_i\}$$

where I_i corresponds to a flag with subspace $V_i = \text{Span}(\{e_\sigma\}_{\sigma \in I_i})$.

Hence,

$$\begin{aligned} \chi(\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)) &= \left| \mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)^{\mathbb{C}^*} \right| \\ &= \binom{n}{r_1, \dots, r_\ell, r_{\ell+1}} \\ &= \frac{n!}{r_1! \dots r_\ell! r_{\ell+1}!} \quad \triangle \end{aligned}$$

4. The Dimension of the Flag Variety.

Lemma 1. *The Grassmannian $G(k, n)$ has (complex) dimension $k(n - k)$*

Proof. The Lie group of $n \times n$ unitary matrices, $U(n)$, acts transitively on \mathbb{C}^n and induces a transitive action on $G(k, n)$. The Grassmannian is a homogeneous space, since the stabilizer is the same for any point in the manifold. We can explicitly determine the stabilizer of $G(k, n)$ by the stabilizer subgroup of $\Lambda = \text{Span}(e_1, \dots, e_k)$.

We first note that the group of automorphisms,

$$\text{Aut}(\Lambda) \cong U(k).$$

Next, let $U_\Lambda \in U(n)$, with columns $\{u_1, \dots, u_n\}$ which must be orthonormal in \mathbb{C}^n .

If U_Λ fixes Λ , then

$$\Lambda = U_\Lambda \cdot \Lambda = \text{Span}(u_1, \dots, u_k)$$

And therefore U_Λ must also fix the subspace $\Lambda' = \text{Span}(e_{k+1}, \dots, e_n)$.

Hence the stabilizer subgroup is isomorphic to $U(k) \times U(n - k)$. So we conclude

$$G(k, n) \cong \frac{U(n)}{U(k) \times U(n - k)}.$$

The latter having (real) dimension $n^2 - k^2 - (n - k)^2 = 2k(n - k)$, since $U(n)$ has (real) dimension n^2 .

Ergo, the Grassmannian $G(k, n)$ has *complex* dimension $k(n - k)$. □

Corollary 1. *The dimension of the Flag Variety, $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$, is*

$$\sum_{i=1}^{\ell-1} k_i(k_{i+1} - k_i) = \sum_{i=1}^{\ell} k_i r_{i+1} = \sum_{1 \leq i < j \leq \ell+1} r_i r_j$$

Proof. Proof goes by induction on ℓ . In the case $\ell = 1$, $\mathcal{F}\ell(k_1, \mathbb{C}^n) \cong G(k_1, n)$ has dimension $k_1(n - k_1)$.

Now, suppose true for some ℓ so that we may prove it for $\ell + 1$. We have the forgetful morphism,

$$\pi_{\ell+1} : \mathcal{F}\ell(k_1, \dots, k_\ell, k_{\ell+1}, \mathbb{C}^n) \rightarrow G(k_{\ell+1}, n)$$

Which only remembers the subspace $V_{\ell+1}$ of the flag V_\bullet . The fibre of each point is isomorphic to $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^{k_{\ell+1}})$.

[G, cor 4.2.7] then says that given a morphism of algebraic varieties, the dimension of the domain is equal to the dimension of the codomain plus the dimension of the fibres, and therefore,

$$\dim \mathcal{F}\ell(k_1, \dots, k_\ell, k_{\ell+1}, \mathbb{C}^n) = \dim G(k_{\ell+1}, n) + \dim \mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^{k_{\ell+1}})$$

$$= k_{\ell+1}(n - k_{\ell+1}) + \sum_{i=1}^{\ell} k_i(k_{i+1} - k_i).$$

Using the convention $n = k_{\ell+2}$, the proof follows by induction. \square

5. The Generalized Plücker Embedding. The Flag Variety $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$ has a natural morphisms to the *Grassmannian variety* $G(k_i, n)$, the space of k_i -subspaces of \mathbb{C}^n , by mapping a flag to its i^{th} subspace.

$$\begin{aligned} \pi_i : \mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n) &\rightarrow G(k_i, n) \\ V_\bullet &\mapsto V_i \end{aligned}$$

Grassmannians are smooth complex projective varieties and $G(k, n)$ can be embedded, via the *Plücker embedding*, into $\mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$.

$$Pl : G(k, n) \hookrightarrow \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$$

$$V = \text{Span}(\{v_1, \dots, v_k\}) \mapsto \bigwedge^k V = \text{Span}(v_1 \wedge \dots \wedge v_k)$$

Such a map is well-defined with respect to choice of spanning vectors up to multiplication by a constant in \mathbb{C}^* and hence well-defined in $\mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$. This map is *equivariant* with respect to the torus action on the Grassmannian, meaning that

$$t \cdot Pl(V) = Pl(t \cdot V), \forall V \in G(k, n),$$

giving the commutative diagram,

$$\begin{array}{ccc} G(k, n) & \xrightarrow{t} & G(k, n) \\ Pl \downarrow & & Pl \downarrow \\ \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right) & \xrightarrow{t} & \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right) \end{array}$$

Hence a subset of the Grassmannian is fixed if and only if its image under the Plücker embedding is fixed.

We now generalize the Plücker embedding to the Flag Variety by the map,

$$\begin{aligned} \mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n) &\hookrightarrow G(k_1, n) \times \dots \times G(k_\ell, n) \hookrightarrow \mathbb{P}\left(\bigwedge^{k_1} \mathbb{C}^n\right) \times \dots \times \mathbb{P}\left(\bigwedge^{k_\ell} \mathbb{C}^n\right) \\ V_\bullet &= \{\emptyset \subset V_1 \subset \dots \subset V_\ell \subset \mathbb{C}^n \mid \dim V_i = k_i\} \mapsto \left(\bigwedge^{k_1} V_1, \dots, \bigwedge^{k_\ell} V_\ell\right) \end{aligned}$$

This embedding inherits equivariance as each of the injections respect the torus action on the Flag Variety.

6. Torus Fixed Curves in the Flag Variety. If C is a genus zero non-singular irreducible projective curve isomorphic to \mathbb{P}^1 without marked points, then we can consider the moduli space of isomorphism classes of maps

$$f : C \rightarrow \mathcal{F}\ell \equiv \mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$$

such that $f_*([C]) = \beta \in H_2(\mathcal{F}\ell)$ for some curve class β in the second homology of $\mathcal{F}\ell$. We call this moduli space $\mathcal{M}_{0,0}(\mathcal{F}\ell, \beta)$. Two maps $(C; f), (C'; f')$ are considered equivalent if there exists

$$\mu \in \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})$$

such that the following diagram commutes,

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathcal{F}\ell \\ \mu \downarrow & \nearrow f' & \\ C' & & \end{array}$$

The torus action on $\mathcal{F}\ell$ defined above induces a torus action on $\mathcal{M}_{0,0}(\mathcal{F}\ell, \beta)$ by,

$$(t \cdot f)(x) = t \cdot f(x), \forall x \in C$$

Since this space admits a torus action, we can calculate its Euler characteristic by enumerating the torus fixed curves of homology class β .

Now, if $f : C \rightarrow \mathcal{F}\ell$ is a torus fixed curve then its image is a rational curve through precisely two torus fixed points [A, Prop 6]. This holds because $f(C)$ is isomorphic to \mathbb{P}^1 , and when we restrict the torus action to $f(C)$ localization implies that, because $\chi(\mathbb{P}^1) = 2$, there are precisely two fixed points which must correspond to two fixed points in $\mathcal{F}\ell$. The image of f is called a torus fixed *line*.

We now arrive at an important result about torus fixed lines in the Flag Variety.

Theorem 2. *Let $\Lambda_\bullet, \Pi_\bullet \in \mathcal{F}\ell$ be two distinct fixed points in the Flag Variety, with corresponding flags of sets I_\bullet, J_\bullet .*

There exists a torus fixed curve containing Λ_\bullet and Π_\bullet if and only if I_\bullet and J_\bullet are related by transposing elements \tilde{I}, \tilde{J}

This last statement means that there exists a *transposition* $\sigma \in S_n$ such that,

$$I_j = \{i_1, \dots, i_{k_j}\} \text{ and } J_j = \{\sigma(i_1), \dots, \sigma(i_{k_j})\}, \text{ for each } j$$

Proof. (\Rightarrow) If there exists a line containing Λ_\bullet and Π_\bullet , then, after composing with the forgetful morphism,

$$\mathcal{F}\ell \xrightarrow{\pi_i} G(k_i, n)$$

the image will be a fixed line through $\Lambda_i, \Pi_i \in G(k_i, n)$. Therefore, due to the main result in [A, §4.3], it is necessary that either,

$$I_i = J_i \text{ or } |I_i \cap J_i| = k_i - 1, \forall i$$

If Λ_\bullet is a fixed point of the Flag Variety, then its image under the Plücker embedding is given by,

$$\Lambda_\bullet \longmapsto ([\varepsilon_{I_1}], \dots, [\varepsilon_{I_\ell}])$$

where $\varepsilon_{I_i} = \bigwedge_{\sigma \in I_i} e_\sigma$ and $[\varepsilon_{I_i}]$ represents its equivalence class in $\mathbb{P}(\bigwedge^{k_i} \mathbb{C}^n)$

The torus fixed line containing $\Lambda_\bullet, \Pi_\bullet$ is then given under the Plücker embedding by,

$$\phi_{IJ}([z : w]) = \left([z^{d_1} \varepsilon_{I_1} + w^{d_1} \varepsilon_{J_1}], \dots, [z^{d_\ell} \varepsilon_{I_\ell} + w^{d_\ell} \varepsilon_{J_\ell}] \right).$$

We want ϕ_{IJ} to be fixed, up to reparametrization, by the torus action.

If $I_i = J_i$ for some i , then $\varepsilon_{I_i} = \varepsilon_{J_i}$. So,

$$[z^{d_i} \varepsilon_{I_i} + w^{d_i} \varepsilon_{J_i}] = [\varepsilon_{I_i}]$$

which is a constant. Hence for the components where $I_i = J_i$ the composition,

$$C \xrightarrow{\phi_{IJ}} \mathcal{Fl} \xrightarrow{\pi_i} G(k_i, n)$$

is a constant (degree zero) map to a torus fixed point in $G(k_i, n)$.

On the other hand, if $|I_i \cap J_i| = k_i - 1$, then we can rewrite,

$$[z \varepsilon_{I_i} + w \varepsilon_{J_i}] = [\varepsilon_{I_i \cap J_i} \wedge (z^{d_i} e_{\tilde{I}_i} + w^{d_i} e_{\tilde{J}_i})]$$

where $\tilde{I}_i \in I_i - J_i$ and similarly with $\tilde{J}_i \in J_i - I_i$ and we note the two are necessarily distinct and uniquely defined.

\mathbb{C}^* acts on this line in the following way,

$$\begin{aligned} t \cdot [z^{d_i} \varepsilon_{I_i} + w^{d_i} \varepsilon_{J_i}] &= [\varepsilon_{I_i \cap J_i} \wedge (z^{d_i} t^{\lambda_{\tilde{I}_i}} e_{\tilde{I}_i} + w^{d_i} t^{\lambda_{\tilde{J}_i}} e_{\tilde{J}_i})] \\ &\equiv [\varepsilon_{I_i \cap J_i} \wedge (z^{d-i} e_{\tilde{I}_i} + w^{d_1} t^{\lambda_{\tilde{J}_i} - \lambda_{\tilde{I}_i}} e_{\tilde{J}_i})] \\ &= \pi_i \circ \phi_{IJ} \circ \mu([z : w]) \end{aligned}$$

via the Möbius transformation $\mu([z : w]) = [z : t^{(\lambda_{\tilde{J}_i} - \lambda_{\tilde{I}_i})/d_i} w]$.

However, the Möbius transform must be *the same* for all i . Furthermore, this must hold for a *general* torus action with *arbitrary* λ_i 's. Therefore we conclude that the \tilde{I}_i, \tilde{J}_i must be the same for all i . In other words I_\bullet and J_\bullet are related by a transposition, in the sense described above. Further the above construction implies non-zero degrees d_i must be the same for all i .

(\Leftarrow) The proof above generalizes to a construction of a torus fixed line given any two flags which are related by a transposition. \square

This result uniquely characterizes which pairs of torus fixed points in $\mathcal{F}\ell$ can be joined by a torus fixed line; they correspond to fixed points whose flags of sets are related by transposing the elements \tilde{I} and \tilde{J} . All that remains is to determine the homology class of the curve given the two fixed points it contains.

7. Degrees and Homology Classes. In general, the Flag Variety has second homology with rank greater than one, and therefore we define the degree of a curve class $\beta \in H_2(\mathcal{F}\ell)$ to be the set of integers, which we refer to as a *multi-index*,

$$i_*(\beta) = d_\bullet = (d_1, \dots, d_\ell)$$

where i is the morphism,

$$i : \mathcal{F}\ell \longrightarrow \mathbb{P} \left(\bigwedge^{k_1} \mathbb{C}^n \right) \times \dots \times \mathbb{P} \left(\bigwedge^{k_\ell} \mathbb{C}^n \right)$$

Given a flag of sets I_\bullet corresponding to a torus fixed point of the Flag Variety. Define

$$I'_i = I_i - I_{i-1}, \text{ so that } I_i = \prod_{j=1}^i I'_j, \text{ and } |I'_i| = r_i, \text{ and each } I'_i \text{ is disjoint from all others.}$$

The purpose of this is to decompose the flag I_\bullet into $(I'_1, \dots, I'_{\ell+1})$, a sequence of disjoint subsets of $\{1, \dots, n\}$. By this definition, we have for each k such that $1 \leq k \leq n$, there is precisely one i , $1 \leq i \leq \ell + 1$, such that $k \in I'_i$.

Suppose $\tilde{I} \in I'_\mu$ and $\tilde{J} \in I'_\nu$ with $\tilde{I} \neq \tilde{J}$. Let J_\bullet be the flag of sets which results from transposing \tilde{I} and \tilde{J} and call each component in the flag J_i . Finally, let ϕ_{IJ} be a morphism from \mathbb{P}^1 into the fixed curve containing the torus fixed points corresponding to I_\bullet, J_\bullet .

If $I'_\mu = I'_\nu$, then $I_\bullet = J_\bullet$ and ϕ_{IJ} is a constant map and hence, $d_i = 0$ for all i .

However in the case, $I'_\mu \neq I'_\nu$, we may assume without loss of generality that $I_\mu \subset I_\nu$, or equivalently, that $\mu < \nu$. The curve will then have non-vanishing degree in all i such that

$$\tilde{I} \in I_i \text{ and } \tilde{J} \notin I_i$$

In these cases $|I_i \cap J_i| = k_i - 1$, so there is a torus fixed line connecting the two.

As we showed in the proof of the previous theorem, the non-vanishing degrees d must all be identical, hence the fixed curves containing the torus fixed points corresponding to I_\bullet, J_\bullet can only have multi-degrees $d \cdot i_*(\beta_{\mu\nu})$ where $i_*(\beta_{\mu\nu}) := (d_1, \dots, d_{\ell-1})$ where,

$$d_j = \begin{cases} 1, & \mu \leq j < \nu \\ 0, & \text{else} \end{cases}$$

This construction also implies that the number of fixed curves of multi-degree $\beta_{\mu\nu}$ containing any torus fixed point in the Flag Variety is precisely $r_\mu r_\nu$.

Remark. If we count all the lines containing a given fixed point, (i.e., all the number of lines of homology class $\beta_{\mu\nu}$ for $\mu < \nu$ containing the given point), we arrive at the sum

$$\sum_{\mu < \nu} r_{\mu} r_{\nu}.$$

Which is equal to the dimension of the Flag Variety. This is no coincidence. Each line containing the given fixed point defines a dimension of the tangent space at that point; we can have no more fixed lines containing a particular point than the dimension of the tangent space which is the same as the dimension of the manifold.

We now have enough enumerative information to compute the Euler characteristic of $M_{0,0}(\mathcal{F}\ell, \beta_{\mu\nu})$. For any fixed point in $\mathcal{F}\ell$, the lines of homology class $\beta_{\mu\nu}$ are determined by fixed points which are obtained by transposing elements $\tilde{I} \in I'_{\mu}$ and $\tilde{J} \in I'_{\nu}$ for some $1 \leq \mu, \nu \leq \ell$ distinct.

There are then $\frac{1}{2}Nr_{\mu}r_{\nu}$ fixed lines of homology class $\beta_{\mu\nu}$ where

$$N = \binom{n}{r_1, \dots, r_{\ell}, r_{\ell+1}} = \chi(\mathcal{F}\ell).$$

We had to divide by two, since we count each fixed line twice – that is, once for each fixed point it contained.

By applying the localization theorem we can conclude,

$$\chi(M(\mathcal{F}\ell, \beta_{\mu\nu})) = \frac{1}{2}Nr_{\mu}r_{\nu}$$

8. The Moduli Stack of Stable Maps. The moduli space introduced in the previous section contains very important enumerative information. However there are problems with working in the space by itself, particularly because it is non-compact. The compactification is achieved through the introduction of stable maps.

For a smooth projective variety X , let $\overline{\mathcal{M}}_{0,m}(X, \beta)$ be the moduli stack of m -pointed, genus zero stable maps

$$f : (C, x_1, \dots, x_m) \rightarrow X, \text{ such that}$$

- (1) C a reduced connected projective *nodal curve* of genus zero, with markings x_1, \dots, x_m and irreducible components $C_i \cong \mathbb{P}^1$.
- (2) The markings x_i are distinct and contained within the smooth locus of C .
- (3) For each irreducible component C_i , the restriction $f_i : C_i \rightarrow X$ is a mapping of degree β_i such that $\sum \beta_i = \beta$.
- (4) For each *collapsed component*, where the restriction f_i has degree zero, the number of nodes plus marked points is at least three.

Two stable maps $(C, x_1, \dots, x_m; f), (C', x_1, \dots, x_m; f')$ are considered equivalent if they are the same up to a reparametrization which identifies each of the marked points.

A *nodal curve* is composed of curves C_1, \dots, C_r isomorphic to \mathbb{P}^1 which are identified at pairs of points $\{p_j, q_j\}$ such that $p_j \in C_{k(j)}$, $q_j \in C_{l(j)}$ for indices $k(j) \neq l(j)$, the identification points are referred to as nodes [K]. This definition makes it easy to define the notion of *the dual graph of C* where each component C_i is represented by a unique vertex and two vertices are connected by an edge if their irreducible components are joined by an identification. Since C has genus zero its dual graph must form a *tree* without cycles. A marked point on C is represented by a *leaf* on the vertex corresponding to the marked component.

Lastly, with regards to the third criterion above, we note that with stable maps to the Flag Variety multi-degrees add component-wise.

Since we are interested in counting torus fixed stable maps, we will adapt the following result from [A].

Proposition 1. *If $[C; f] \in \overline{\mathcal{M}}_{0,0}(G(k, n), d)$ is a torus fixed stable map and $x \in C$ is a node, then $f(x)$ is a torus fixed point of $G(k, n)$*

This easily generalizes to the Flag Variety since, if $[C; f] \in \overline{\mathcal{M}}_{0,0}(\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n), \beta)$ and $x \in C$ is a node, then $(\pi_i \circ f)(x)$ must be a fixed point in $G(k_i, n)$, $\forall i$, or equivalently,

$$(\pi_i \circ f)(x) = V_{I_i} = \text{Span}(\{e_\sigma\}_{\sigma \in I_i}) \text{ for some } I_i \subset [n] \text{ such that } |I_i| = k_i, \forall i$$

which implies that $f(x)$ is a fixed point in the Flag Variety.

9. Calculations. For our first few calculations we shall consider a Flag Variety with two partitions of \mathbb{C}^n ,

$$X = \mathcal{F}\ell(k_1, k_2, \mathbb{C}^n),$$

and calculate the number of torus fixed stable maps with zero markings in genus zero for degrees $\beta_{1,2}$; $\beta_{2,3}$; $\beta_{1,3}$; and $\beta_{1,3} + \beta_{2,3}$. The localization theorem then implies that for any curve class $\beta \in H_2(\mathcal{X})$, the number of fixed maps is equal to the Euler characteristic, $\chi(\overline{\mathcal{M}}_{0,0}(X, \beta))$. For $\beta_{1,2}$; $\beta_{2,3}$; and $\beta_{1,3}$ we also calculate the Euler characteristics in the case of η marked points for $\eta \geq 3$.

Example 2. For $\beta_{1,2}$ and $\beta_{2,3}$ ($i_*(\beta_{1,2}) = (1, 0)$, and $i_*(\beta_{2,3}) = (0, 1)$) the only stable maps are simply the rational maps themselves. Hence we get,

$$\chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{1,2})) = \frac{1}{2}Nr_1r_2, \text{ and } \chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{2,3})) = \frac{1}{2}Nr_2r_3$$

We can also compute the Euler characteristic for stable maps with $\eta \geq 3$ markings. Define

$$\Psi(q) = \sum_{\eta=2}^{\infty} \frac{q^\eta}{\eta!} \chi(\overline{\mathcal{M}}_{0,\eta+1})$$

where $\overline{\mathcal{M}}_{0,\eta+1} := \overline{\mathcal{M}}_{0,\eta+1}(\{pt.\}, 0)$. We will use the convention $\Omega(\eta + 1) := \chi(\overline{\mathcal{M}}_{0,\eta+1})$. Manin proved in [M], that this formal sum is the unique solution to the equation,

$$(1 + q + \Psi(q))\log(1 + q + \Psi(q)) = 2\Psi(q) + q$$

We can expand this as a power series and, in principle, determine all of the coefficients of $\Psi(q)$.

We now compute the Euler characteristic of $\overline{\mathcal{M}}_{0,\eta}(X, \beta)$ for $\beta = \beta_{1,2}$ as a generating series.

Lemma 2.

$$(1) \quad \Phi(q) := \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \chi(\overline{\mathcal{M}}_{0,\eta}(X, \beta_{1,2}))$$

$$(2) \quad = \frac{1}{2} N r_1 r_2 (2\Psi(q) - q^2 + 2q\Psi(q) + \Psi(q)^2)$$

Proof. We need to count the number of fixed stable maps with η markings. Since the stable maps must be fixed, the map must take the markings to a fixed point of \mathcal{X} . So pick any fixed curve of degree $\beta_{1,2}$. Such a curve contains precisely two fixed points, we can therefore attach one tree, T_1 of collapsed components to one of the fixed points, and another tree, T_2 to the other fixed point. We make a simplification by defining the constant, $e = \frac{1}{2} N r_1 r_2$.

Case 1, $|T_1| = \eta$ and $|T_2| = 0$, by symmetry this is the same as the number of cases where $|T_2| = \eta$ and $|T_1| = 0$. The number of ways to attach a tree with η marked points to a fixed point on a curve is $\Omega(\eta + 1)$ (we add one to count the attaching point which must be kept track of). Hence there are $2e\Omega(\eta + 1)$ fixed maps of this form, where $e := \frac{1}{2} N r_1 r_2$.

Case 2, $|T_1| = \eta - 1$ and $|T_2| = 1$. Since we must choose which of the η marked points gets mapped to T_2 the total contribution is $2e\eta\Omega(\eta)$ where we have counted twice to account for symmetry.

Case 3, $|T_1| = \eta_1$ and $|T_2| = \eta_2$ where $\eta_1 + \eta_2 = \eta$, and $\eta_1, \eta_2 \geq 2$ for stability. This gives a total of $e\Omega(\eta_1 + 1)\Omega(\eta_2 + 1)$.

Now we put these into a generating series,

$$(3) \quad \Phi(q) := \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \chi(\overline{\mathcal{M}}_{0,\eta}(X, \beta_{1,2}))$$

$$(4) \quad = e(2 \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \Omega(\eta + 1) + 2 \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \eta \Omega(\eta) + \sum_{\eta_1, \eta_2 \geq 3} \frac{q^\eta}{\eta_1! \eta_2!} \Omega(\eta_1 + 1) \Omega(\eta_2 + 1))$$

$$(5) \quad = e(2\Psi(q) - q^2 + 2q\Psi(q) + \Psi(q)^2)$$

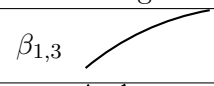
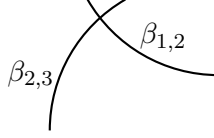
	Strata Diagrams	Number
(i)		$\frac{1}{2}Nr_1r_3$
(ii)		$Nr_1r_2^2r_3$

TABLE 1. The two strata for $\overline{\mathcal{M}}_{0,0}(X, \beta_{1,3})$

Note that we had to subtract $q^2\Omega(3)$ from $\Psi(q)$ to get the power series to match the first sum. We then used $\Omega(3) = 1$ since there is only one stable map with three markings in $\overline{\mathcal{M}}_{0,3}$. \square

If we wanted to compute the series for $\overline{\mathcal{M}}_{0,\eta}(X, \beta_{2,3})$ all we would need to do is replace $e = \chi(\overline{\mathcal{M}}_{0,\eta}(X, \beta_{2,3}))$.

Example 3. Next we shall compute the number of fixed stable maps for the case $\beta_{1,3}$ ($i_*(\beta_{1,3}) = (1, 1)$) at first with no marked points and then with marked points added in. By the localization theorem this is then equal to

$$\chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{1,3})).$$

There are two nodal curves, C , to consider which we include in the Table 1.

In the first case, we can pick any fixed curve with degree $\beta_{1,3}$ of which there are

$$\frac{1}{2}Nr_1r_3.$$

To count the number of such diagrams in the second case, we can pick any fixed point for the node to coincide with, and then pick one line of degree $\beta_{1,2}$ and one line of degree $\beta_{2,3}$ which pass through the fixed point, giving

$$N(r_1r_2)(r_2r_3)$$

The Euler characteristic is the sum of these two giving,

$$\chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{1,3})) = \frac{1}{2}Nr_1r_3(1 + 2r_2^2).$$

We can now add $\eta \geq 3$ marked points and compute the Euler characteristics. We state the result of our calculation as a lemma.

Lemma 3.

$$(6) \quad \Phi(q) := \sum_{\eta=3}^{\infty} \chi(\overline{\mathcal{M}}_{0,\eta}(\mathcal{X}, \beta_{1,2}))$$

$$(7) \quad = \frac{1}{2} N r_1 r_3 (1 + 2r_2^2) (2(q+1)\Psi(q) + \Psi(q)^2 - q^2)$$

$$(8) \quad + N r_1 r_2^2 r_3 (\Psi(q)^2 \Psi'(q) + 2(q+1)\Psi(q)\Psi'(q) + \Psi'(q)(q+1)^2 - 3q^2 - q)$$

Proof. By Lemma 2 above, the contribution of diagrams of type (i) in Table 1 is,

$$\frac{1}{2} N r_1 r_3 (2\Psi(q) - q^2 + 2q\Psi(q) + \Psi(q)^2).$$

What remains is to count the contribution by diagrams of type (ii) after adding η marked points.

Let T_0 be the tree of collapsed components which map to the fixed point in the intersection of the two curves, this tree has two attaching points: one on the curve of degree $\beta_{2,3}$ and one of curve of degree $\beta_{1,2}$, hence T_0 needs only one marked point for stability. Let T_1 be the tree of collapsed components which is mapped to the other fixed point on the curve of degree $\beta_{2,3}$ and T_2 be the tree of collapsed components which map to the remaining fixed point on the curve of degree $\beta_{1,2}$.

We divide the calculation into three parts. First, those loci which satisfy $|T_0| = 0$. Next, the remaining loci which satisfy $|T_1|, |T_2| \leq 1$. Finally, those loci which satisfy at least one of, $|T_1| \geq 2$, or $|T_2| \geq 2$. It can be checked that any torus fixed stable map with $\eta \geq 3$ markings must fall into one of these categories, and hence this list is exhaustive. The generating series is then given by the sum of these contributions plus the contribution from the strata of type (i).

First, if $|T_0| = 0$, then the combinations are the same as in the previous example. The contributions due to this condition give

$$e'(2\Psi(q) - q^2 + 2q\Psi(q) + \Psi(q)^2), \text{ where } e' = N r_1 r_2^2 r_3.$$

The variable, e' , is just the number of lines of the form (ii) in Table 1.

Next, if both $|T_1| = |T_2| = 0$ we have a contribution of $e'\Omega(\eta+2)$. If $|T_1| = 0, |T_2| = 1$, we get $2e'\eta\Omega(\eta+1)$ (also counting fixed points where the conditions on T_1, T_2 are

reversed). If both $|T_1| = |T_2| = 1$, we have $\eta(\eta - 1)e'\Omega(\eta)$. Together these contribute,

$$(9) \quad e' \left(2 \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \eta \Omega(\eta + 1) + \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \Omega(\eta + 2) + \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \eta(\eta - 1) \Omega(\eta) \right)$$

$$(10) \quad = e' \left(2q \sum_{\eta=2}^{\infty} \frac{q^\eta}{\eta!} \Omega(\eta + 2) + \sum_{\eta=3}^{\infty} \frac{q^\eta}{\eta!} \Omega(\eta + 2) + q^2 \sum_{\eta=1}^{\infty} \frac{q^\eta}{\eta!} \Omega(\eta + 2) \right)$$

$$(11) \quad = e' \left(2q(\Psi'(q) - q\Omega(3)) + (\Psi'(q) - \frac{q^2}{2}\Omega(4) - q\Omega(3)) + q^2\Psi'(q) \right)$$

$$(12) \quad = e'(\Psi'(q)(q + 1)^2 - 3q^2 - q)$$

$$\text{where } \Psi'(q) = \frac{\partial \Psi}{\partial q} = \sum_{\eta=1}^{\infty} \frac{q^\eta}{\eta!} \Omega(\eta + 2)$$

In the last step we used that $\Omega(3) = 1$, and, since $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, that

$$\Omega(4) = \chi(\overline{\mathcal{M}}_{0,4}) = 2.$$

For the remain three cases, first suppose $|T_1| = \eta_1 \geq 2$, and $|T_2| = 0$, we then have $|T_0| = \eta_0$ such that $\eta_0 + \eta_1 = \eta$. We have to pick η_0 of the markings to be in T_0 hence the contribution from strata of this form is, $2e' \binom{\eta}{\eta_0} \Omega(\eta_1 + 1) \Omega(\eta_0 + 2)$. Next, suppose $|T_2| = 1$ so that $\eta_1 + \eta_0 + 1 = \eta$ contributing $2e' \binom{\eta}{\eta_0, \eta_1, 1} \Omega(\eta_1 + 1) \Omega(\eta_0 + 2)$. Finally, for arbitrary $|T_0|, |T_1|, |T_2| \geq 2$ such that $\eta_0 + \eta_1 + \eta_2 = \eta$ we get a contribution of $e' \binom{\eta}{\eta_0, \eta_1, \eta_2} \Omega(\eta_1 + 1) \Omega(\eta_0 + 2) \Omega(\eta_2 + 1)$. Giving a total contribution of,

$$(13) \quad e' \sum_{\eta=2}^{\infty} \left(\sum_{\eta_1 + \eta_2 + \eta_0 = \eta} \frac{q^\eta}{\eta_1! \eta_2! \eta_0!} \Omega(\eta_1) \Omega(\eta_2) \Omega(\eta_0) \right) + 2e' \sum_{\eta=2}^{\infty} \left(\sum_{\eta_1 + \eta_0 + 1 = \eta} \frac{q^\eta}{\eta_1! \eta_0! 1!} \Omega(\eta_1) \Omega(\eta_0) \right)$$

$$(14) \quad + 2e' \sum_{\eta=2}^{\infty} \left(\sum_{\eta_1 + \eta_0 = \eta} \frac{q^\eta}{\eta_1! \eta_0!} \Omega(\eta_1 + 1) \Omega(\eta_0 + 2) \right)$$

$$(15) \quad = e'(\Psi(q)^2 \Psi'(q) + 2q\Psi(q)\Psi'(q) + 2\Psi(q)\Psi'(q))$$

The sum of all four of these terms gives us our final generating series,

$$(16) \quad \Phi(q) := \sum_{\eta=3}^{\infty} \chi(\overline{\mathcal{M}}_{0,\eta}(\mathcal{X}, \beta_{1,2}))$$

$$(17) \quad = \frac{1}{2} N r_1 r_3 (1 + 2r_2^2) (2(q + 1)\Psi(q) + \Psi(q)^2 - q^2)$$

$$(18) \quad + N r_1 r_2^2 r_3 (\Psi(q)^2 \Psi'(q) + 2(q + 1)\Psi(q)\Psi'(q) + \Psi'(q)(q + 1)^2 - 3q^2 - q)$$

	Strata Diagram	Number
(i)		$N(r_1 r_3)(r_2 r_3)$
(ii)		$N(r_1 r_2)(r_2 r_3)$
(iii)		$(\frac{1}{2} N r_1 r_2)(2 r_2 r_3)(r_2 r_3) = N r_1 r_2^3 r_3^2$
(iv)		$(N r_1 r_2)(2 r_2 r_3)(r_2 r_3) = N r_1 r_2^3 r_3^2$
(v)		$(N r_1 r_2) \frac{(r_2^2 r_3^2 + r_2 r_3)}{2}$

 TABLE 2. The five types of strata for $\overline{\mathcal{M}}_{0,0}(X, \beta_{1,3} + \beta_{2,3})$

□

Example 4. Our next calculation is more involved. Consider stable maps of degree $\beta_{1,3} + \beta_{2,3}$ with no markings ($i_*(\beta_{1,3}) + i_*(\beta_{2,3}) = (1, 2)$). There are five strata to consider (see Table 2).

Summing up all the contributions enumerated in Table 2, we then have the total as,

$$\chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{1,3} + \beta_{2,3})) = \frac{1}{2} N r_1 r_2 (5 r_2^2 r_3^2 + 2 r_2 r_3 + r_3^2)$$

Cases (i) and (ii) are similar to the calculation in the previous example. In case (iii), pick one line of degree $\beta_{1,2}$ ($\frac{1}{2} N r_1 r_2$ ways) and then pick one of the $2 r_2 r_3$ lines of degree $\beta_{2,3}$ which contain either of the two nodes, and then pick another line which contains the second node ($r_2 r_3$ possible choices). Case (iv) is only slightly different; pick a line of degree $\beta_{1,2}$ then pick a line of degree $\beta_{2,3}$ which contain either of the two nodes, then pick one of the lines of degree $\beta_{2,3}$ which intersects the other node of the second line. For

case (v), we must first pick one of the N fixed points to map the collapsed component to, then pick a line of degree $\beta_{1,2}$ which contains it, then we must pick two of the lines of degree $\beta_{2,3}$ which contain it. we can either choose two different lines to map them to, or we can choose to map them to the same line, giving

$$\binom{r_2 r_3}{2} + r_2 r_3 = \frac{r_2^2 r_3^2 + r_2 r_3}{2}.$$

If we wanted to calculate the Euler characteristic with markings, the methods used above could in principal be extended to the strata above with little difficulty.

△

Example 5. Now let's examine a more complicated space. Let $X = \mathcal{F}\ell(k_1, k_2, k_3, k_4, \mathbb{C})$. We will calculate $\chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{1,4}))$ where $i_*(\beta_{1,4}) = (1, 1, 1)$. We again give a table of all possible strata (Table 3), the justifications are similar to those given in the previous example. Counting over all the possible fixed points we arrive at our final answer,

$$(19) \quad \chi(\overline{\mathcal{M}}_{0,0}(X, \beta_{1,4})) = \frac{1}{2} N r_1 r_4 (1 + 2r_2^2 2r_3^2 + 4r_2^2 r_3^2)$$

$$(20) \quad = \frac{1}{2} N r_1 r_4 (1 + 2r_2^2)(1 + 2r_3^2)$$

Note the similarity of this Euler characteristic to that of Example 3. It is the opinion of the author that this is not a coincidence, he conjectures that this pattern would continue for Flag Varieties parametrizing flags of arbitrary length. Although the methods used above could, in principle, be extended to flags of arbitrary length, the combinatorial difficulty increases rapidly with the number of flags. Thus a proof of this may be beyond the methods of this paper.

	Strata Diagram	Number
(i)		$\frac{1}{2}Nr_1r_4$
(ii)		$N(r_1r_3)(r_3r_4)$
(iii)		$N(r_1r_2)(r_2r_4)$
(iv)		$(\frac{1}{2}Nr_1r_2)(2r_2r_3)(r_3r_4) = Nr_1r_2^2r_3^2r_4$
(v)		$(\frac{1}{2}Nr_1r_2)(2r_2r_3)(r_3r_4) = Nr_1r_2^2r_3^2r_4$
(vi)		$(\frac{1}{2}Nr_1r_2)(2r_2r_3)(r_3r_4) = Nr_1r_2^2r_3^2r_4$
(vii)		$N(r_1r_2)(r_2r_3)(r_3r_4)$

TABLE 3. The seven types of strata for $\overline{\mathcal{M}}_{0,0}(X, \beta_{1,4})$

APPENDIX A. THE POINCARÉ POLYNOMIAL OF THE FLAG VARIETY

The *quantum binomial*, also called q-binomial coefficient, is an analogue of the binomial coefficient. It is defined as,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

where $[n]_q = \frac{1-q^n}{1-q}$, and $[n]_q! = [1]_q[2]_q \dots [n]_q$. The quantum binomial comes up naturally in the cohomology of the Grassmannian.

Proposition 2. *The Poincaré polynomial of $G(k, n)$, $P_{G(k, n)}(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$*

We prove this using the Białynicki-Birula Localization Theorem. There are other methods of calculating the Poincaré polynomial which can be found in the literature on the cohomology of the Grassmannian.

Theorem 3. *(Białynicki-Birula Localization Theorem) If a smooth manifold X admits a \mathbb{C}^* action with fixed points p_1, \dots, p_e , then the Poincaré polynomial of X is given by*

$$P_X(q) = \sum_{i=1}^e q^{2d_i^+}$$

where d_i^+ is the number of positive weights on the tangent space of X at the point p_i

We now prove the proposition.

Proof. Let \mathbb{C}^* act on \mathbb{C}^n with weights $\lambda_1 < \dots < \lambda_n$, inducing action on points of $G(k, n)$. If $V \subset \mathbb{C}^n$ is k -dimensional subspace,

$$t \cdot V = \{t \cdot v | v \in V\}$$

The torus action on $G(k, n)$ has $\binom{n}{k}$ fixed points,

$$\Lambda_I = \text{Span}(e_{i_1}, \dots, e_{i_k}) \text{ for all } I = \{i_1 < \dots < i_k\} \subset [n] := \{1, \dots, n\}$$

Given a fixed point Λ_I the tangent space of $G(k, n)$ at Λ_I is standardly known to be,

$$T_{\Lambda_I} G(k, n) = \text{Hom}(\Lambda_I, \mathbb{C}^n / \Lambda_I)$$

Lemma 4. *If \mathbb{C}^* acts on V with weights $\alpha_1, \dots, \alpha_k$ and \mathbb{C}^* acts on W with weights $\beta_1, \dots, \beta_\ell$, then \mathbb{C}^* acts on $\text{Hom}(V, W)$ and the weights are $\beta_j - \alpha_i$*

The proof of the lemma uses that $\text{Hom}(V, W) = V^* \otimes W$ and hence the weights of the torus action on the dual V^* are negated.

Noting for a fixed point $\Lambda_I \in G(k, n)$, the torus action on \mathbb{C}^n induces a torus action on the linear subspace Λ_I with weights $\lambda_{i_1} < \dots < \lambda_{i_k}$ and a torus action on \mathbb{C}^n / Λ_I with the other $n - k$ weights.

To apply the localization theorem, we need to determine the number of *positive* weights using the previous lemma. For each $i_\mu \in I$, there are $n - k - i_\mu + \mu$ weights on \mathbb{C}^n / Λ_I strictly larger than λ_{i_μ} .

Thus the number of positive weights on $\text{Hom}(\Lambda_I, \mathbb{C}^n / \Lambda_I)$ is,

$$d_I^+ = \sum_{j=1}^k n - k + j - i_j$$

$$= k(n-k) + \frac{k(k+1)}{2} - \sum_{j=1}^k i_j$$

The localization theorem then states that,

$$P_{G(k,n)}(q) = \sum_{I \subset [n], |I|=k} q^{d_I^+}$$

Lemma 5.

$$\sum_{I \subset [n], |I|=k} q^{d_I^+} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

where $d_I^+ = k(n-k) + \frac{k(k+1)}{2} - \sum_{i_j \in I} i_j$.

Proof. The proof goes by induction. Start with the case $n = 1$. If $k = 0, 1$ there is only a single subset of $[n]$ with cardinality k . A calculation yields $d_I^+ = 0$ and

$$\sum_{I \subset [n], |I|=k} q^{d_I^+} = 1 = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

If $k > n$ then the claim is vacuous as both sides of the equation vanish. Hence, for $n = 1$ the claim holds for all non-negative k . Now, suppose for the purpose of induction that the lemma holds for some n and all non-negative k . Then for the case $n + 1$ we can separate the sum into those terms where $(n + 1) \notin I \subset [n + 1]$ and those terms where $(n + 1) \in I \subset [n + 1]$. In the later case we can define $I = I' \cup \{n + 1\}$ where $I' \subset [n]$ and $|I'| = k - 1$. We then have

$$\sum_{I \subset [n+1], |I|=k} q^{d_I^+} = \sum_{I \subset [n], |I|=k} q^{d_I^+} + \sum_{I' \subset [n], |I'|=k-1} q^{d_{I'}^+}$$

We can reduce the exponent in the second term as follows,

$$(21) \quad d_{I'}^+ \Big|_{|I'|=k-1} = (k-1)(n-k+1) + \frac{k(k-1)}{2} - \sum_{i_j \in I'} i_j$$

$$(22) \quad = k(n-k) - (n-k) + k-1 + \frac{k(k+1)}{2} - k - \sum_{i_j \in I'} i_j$$

$$(23) \quad = k(n-k) + \frac{k(k+1)}{2} - \sum_{i_j \in I'} i_j - (n+1) - (n-k+1)$$

$$(24) \quad = d_{I'}^+ \Big|_{I=I' \cup \{n+1\}} - (n-k+1)$$

So by the induction hypothesis we arrive at

$$(25) \quad \sum_{I \subset [n+1], |I|=k} q^{d_I^+} = \sum_{I \subset [n], |I|=k} q^{d_I^+} + \sum_{I' \subset [n], |I'|=k-1} q^{d_{I'}^+}$$

$$(26) \quad = \sum_{I \subset [n], |I|=k} q^{d_I^+} + q^{(n+1-k)} \left(\sum_{I' \subset [n], |I'|=k-1} q^{d_{I'}^+} \right)$$

$$(27) \quad = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{(n+1-k)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

This reduces the proof to the following q-binomial coefficient identity.

Lemma 6. $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{(n+1-k)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$

Proof.

$$(28) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{(n+1-k)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} + q^{(n+1-k)} \frac{[n]_q!}{[k-1]_q! [n+1-k]_q!}$$

$$(29) \quad = \frac{[n]_q!}{[k]_q! [n+1-k]_q!} ([n+1-k]_q + q^{(n+1-k)} [k]_q)$$

$$(30) \quad = \frac{[n]_q!}{[k]_q! [n+1-k]_q!} ([n+1]_q) = \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$$

The last line follows from an application of the definitions,

$$(31) \quad [n+1-k]_q + q^{(n+1-k)} [k]_q = \frac{q^{(n+1-k)} - 1}{q-1} + q^{(n+1-k)} \frac{q^k - 1}{q-1} = \frac{q^{(n+1-k)} - 1}{q-1} + \frac{q^{n+1} - q^{(n+1-k)}}{q-1}$$

$$(32) \quad = \frac{q^{n+1} - 1}{q-1} = [n+1]_q$$

which completes the induction argument for Lemma 5 and hence,

$$\sum_{I \subset [n], |I|=k} q^{d_I^+} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Ergo,

$$P_{G(k,n)}(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

which completes the proof. \square

Corollary 2. *The Poincaré polynomial of the Flag Variety $\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)$ is given by,*

$$P_{\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)}(q) = \begin{bmatrix} n \\ r_1, \dots, r_{\ell+1} \end{bmatrix}_q$$

Proof. The first step is given by the Poincaré polynomial of the Grassmannian,

$$P_{G(k_1, k_2)}(q) = \begin{bmatrix} k_2 \\ k_1 \end{bmatrix}_q = \frac{[k_2]!_q}{[k_1]!_q [k_2 - k_1]!_q} = \begin{bmatrix} k_2 \\ r_1, r_2 \end{bmatrix}_q$$

The proof then follows inductively from the fibration,

$$\begin{array}{ccc} \mathcal{F}\ell(k_1, \dots, k_{\ell-1}, \mathbb{C}^{k_\ell}) & \longrightarrow & \mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n) \\ & & \downarrow \\ & & G(k_\ell, n) \end{array}$$

The Poincaré polynomial is multiplicative with respect to fibrations of smooth varieties with smooth fibres, and therefore

$$\begin{aligned} P_{\mathcal{F}\ell(k_1, \dots, k_\ell, \mathbb{C}^n)}(q) &= P_{\mathcal{F}\ell(k_1, \dots, k_{\ell-1}, \mathbb{C}^{k_\ell})}(q) P_{G(k_\ell, n)}(q) \\ &= \begin{bmatrix} k_\ell \\ r_1, \dots, r_\ell \end{bmatrix}_q \begin{bmatrix} n \\ k_\ell \end{bmatrix}_q \\ &= \begin{bmatrix} n \\ r_1, \dots, r_\ell, r_{\ell+1} \end{bmatrix}_q \end{aligned}$$

□

REFERENCES

- [A] S. Agrawal, *The Euler Characteristic of the Moduli Space of Stable Maps in a Grassmannian*. June 2011.
- [G] A. Gathmann, *Algebraic Geometry: Notes for a class taught at the University of Kaiserslautern 2002/2003*. Available at <http://www.mathematik.uni-kl.de/~gathmann/class/algeom-2002/main.pdf>.
- [L] A. López-Martín, *Poincaré Polynomials of Stable Map Spaces to Grassmannians*. 2013. Pre-print available at <http://alopez.math.tufts.edu/research/20130312LM-Poincare.pdf>.
- [K] S. Katz, *Enumerative Geometry and String Theory*. American Mathematical Society, 2006.
- [M] Y. Manin, *Generating Functions in Algebraic Geometry and sums over trees*. eprint arXiv:alg-geom/9407005, 1994.
- [O] D. Oprea, *Tautological classes on the moduli spaces of stable maps to \mathbb{P}^r via torus actions*. Adv. Math., 207(2):661-690, 2006.

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