Chapter 8

<u>Defn</u>: Let G_1 and G_2 be two groups.
The <u>(external) direct product</u> of G_1 and G_2 is the group

$$G_1 \times G_2 = \{(a_1, a_2) : a_1 \in G_1, a_2 \in G_2\}$$

with operation

$$(a_1, a_2)(b_1, b_2) = (\underbrace{a_1 b_1}_{\text{operation in } G_1}, \underbrace{a_2 b_2}_{\text{operation in } G_2})$$

<u>Ex</u>: $\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2), (0,3), (1,3)\}$
and $(a,b) + (c,d) = (a+c, b+d)$

operation in $\mathbb{Z}_2$, so $a+c \bmod 2$

operation in $\mathbb{Z}_4$, so $b+d \bmod 4$

So $(1,2)+(1,3)=(0,1)$

<u>Ex</u>: $U(4) \times U(5) = \{(1,1), (3,1), (1,2), (3,2), (1,3), (3,3), (1,4), (3,4)\}$
and $(3,2)(3,4) = (1,3)$

$3 \cdot 3 \bmod 4 = 1$

$2 \cdot 4 \bmod 5 = 3$

Similarly, for groups $G_1, \cdots, G_n$ we can define

$$G_1 \times G_2 \times \cdots \times G_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in G_1, a_2 \in G_2, \cdots, a_n \in G_n\}$$

with operation $(a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n) = (a_1 b_1, a_2 b_2, \cdots, a_n b_n)$

<u>Thm 8.1</u>: The order of an element $(g_1, g_2, \cdots, g_n) \in G_1 \times G_2 \times \cdots \times G_n$ is lcm(order of g_1, order of g_2, $\cdots$, order of g_n)

Ex: $\mathbb{Z}_6 \times \mathbb{Z}_8$ and $\mathbb{Z}_{48}$ are both groups of order 48, but they are not isomorphic:

$\mathbb{Z}_{48}$ has an element of order 48

but $\mathbb{Z}_6 \times \mathbb{Z}_8$ does not.

Ex: Find all elements of $\mathbb{Z}_{12} \times \mathbb{Z}_9$ that have order 6.

The elements of $\mathbb{Z}_{12}$ have orders 1, 2, 3, 4, 6, 12 ①

The elements of $\mathbb{Z}_9$ have orders 1, 3, 9 ②

We need to find numbers from row ① and numbers from row ② whose lcm is 6:

$$6 = \mathrm{lcm}(2,3) = \mathrm{lcm}(6,1) = \mathrm{lcm}(6,3)$$

Elements in $\mathbb{Z}_{12}$ having order 2:	Elements in $\mathbb{Z}_{12}$ having order 6:	Elements in $\mathbb{Z}_9$ having order 3:	Elements in $\mathbb{Z}_9$ having order 1:
6	2, 10	3, 6	0

The elements of $\mathbb{Z}_{12} \times \mathbb{Z}_9$ having order 6 are:

(6, 3)
(6, 6) — lcm(2, 3)

(2, 0)
(10, 0) — lcm(6, 1)

(2, 3)
(2, 6)
(10, 3)
(10, 6) — lcm(6, 3)

Thm 8.2 Let G_1, G_2 ~~G_1, G_2~~ be finite cyclic groups. Then $G_1 \times G_2$ is cyclic if and only if $\gcd(|G_1|, |G_2|) = 1$.

Cor 2: If $n_1, n_2, \cdots, n_k$ are relatively prime then
$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \approx \mathbb{Z}_{n_1 n_2 \cdots n_k}$$

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_3 \approx \mathbb{Z}_6$

In fact $\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1,1) \rangle$ and $(1,1)$ has order 6.