HW 6 due Wed.
OH today: 11:00–11:30, 4:00–4:30

Chapter 7

<u>Notation</u>: For a group G, $S \subseteq G$, $a \in G$ define

$aS = \{as : s \in S\}$	$a+S$
$Sa = \{sa : s \in S\}$	$S+a$
$aSa^{-1} = \{asa^{-1} : s \in S\}$	in additive notation

$|S|$ = number of elements in S

<u>Defn</u>: Let G group, $H \leq G$ subgroup, $a \in G$.

- aH is the <u>left-coset of H containing a</u>
  a is called a <u>coset representative</u> of aH.
- Ha is the <u>right-coset of H containing a</u>
  a is called a <u>coset representative of Ha</u>.

Lem. 7.A: Let G group, $H \leq G$ (subgroup), $a, b \in G$.

Then $aH \overset{①}{=} bH \iff a \overset{②}{\in} bH \iff b^{-1}a \overset{③}{\in} H$

Furthermore, either $aH = bH$ or $aH \cap bH = \emptyset$.

(Similarly $Ha = Hb \iff a \in Hb \iff ab^{-1} \in H$
and either $Ha = Hb$ or $Ha \cap Hb = \emptyset$)

Pf: (① ⇒ ②) Assume $aH = bH$. Since $e \in H$ we have

$$a = ae \in aH = bH$$

(② ⇒ ③) Assume $a \in bH$. Then there is $h \in H$ with $a = bh$. So $b^{-1}a = h \in H$.

(③ ⇒ ①) Assume $b^{-1}a \in H$. Set $h_0 = b^{-1}a \in H$.
Notice $h_0^{-1} = a^{-1}b$.

($aH \subseteq bH$) For any $h \in H$ we have

$$ah = (bb^{-1})ah = b(b^{-1}a)h = b\underbrace{h_0 h}_{\text{since } h_0 h \in H} \in bH$$

($bH \subseteq aH$) For any $h \in H$ we have

$$bh = (aa^{-1})bh = a(a^{-1}b)h = a\underbrace{h_0^{-1} h}_{\text{since } h_0^{-1} h \in H} \in aH$$

Now we prove the "Furthermore".

Case 1: $aH \cap bH = \emptyset$ Done.

Case 2: $aH \cap bH \neq \emptyset$.

Pick any $c \in aH \cap bH$.
Then $c \in aH$ and $c \in bH$.
By ② ⇒ ①, $cH = aH$ and $cH = bH$.
Therefore $aH = bH$. $\square$

Lem. 7.B: The collection of left-cosets $\{aH : a \in G\}$ partitions G. Also $|aH| = |H|$ for all $a \in G$.
(Similarly $\{Ha : a \in G\}$ partitions G and $|Ha| = |H|$ for all $a \in G$)

Pf: Since $e \in H$, we have $a = ae \in aH$. So the union of the aH $(a \in G)$ is equal to G. By lem 7.A the sets aH $(a \in G)$ are disjoint when they are not equal. This shows that $\{aH : a \in G\}$ is a partition of G.

Lastly, $|aH| = |H|$ since the map from H to aH sending $h \in H$ to $ah \in aH$ is one-to-one and onto. $\square$

Warning: Generally, $aH \neq Ha$. However...

Lem 7.C: $aH = Ha \Leftrightarrow aHa^{-1} = H$

Pf: Multiplication on the right by a^{-1} is a one-to-one operation that sends aH to aHa^{-1} and sends Ha to H. $\square$

Ex: Set $H = \{\alpha \in S_3 : \alpha(1) = 1\} = \{\varepsilon, (23)\}$.
H subgroup of S_3.

The left cosets of H are
$\varepsilon H = H = \{\varepsilon, (23)\} = (23)H = \{\alpha \in S_3 : \alpha(1) = 1\}$
$(12)H = \{(12), (123)\} = (123)H = \{\alpha \in S_3 : \alpha(1) = 2\}$
$(13)H = \{(13), (132)\} = (132)H = \{\alpha \in S_3 : \alpha(1) = 3\}$

The right cosets of H are
$H\varepsilon = H = \{\varepsilon, (23)\} = H(23) = \{\alpha \in S_3 : \alpha(1) = 1\}$
$H(12) = \{(12), (132)\} = H(132) = \{\alpha \in S_3 : \alpha(2) = 1\}$
$H(13) = \{(13), (123)\} = H(123) = \{\alpha \in S_3 : \alpha(3) = 1\}$

Lagrange's Thm 7.1:

If G is a finite group and $H \leq G$ is a subgroup then $|H|$ divides $|G|$. Moreover the number of distinct left (or right) cosets of H in G is $|G|/|H|$

Called the index of H in G and is denoted $|G:H|$.

Pf: Let $r = |G:H|$ = # of distinct left cosets of H.

Let $a_1H, a_2H, \ldots, a_rH$ be the distinct left cosets of H. Then by Lem 7.B $a_1H, \ldots, a_rH$ partition G so

$$|G| = |a_1H| + |a_2H| + \cdots + |a_rH|$$

$$\overset{\text{Lem 7.B}}{=} \underbrace{|H| + |H| + \cdots + |H|}_{r}$$

$$= r \cdot |H|.$$

Therefore $|H| \mid |G|$ and

$$|G:H| = r = |G|/|H|.$$

$\square$