HW 6 due Wed

OH Wed: 11:00–11:30, 4:00–4:30

Defn: Given a seq (a_n) in $\mathbb{C}$ we write (for $p \leq q$) $\sum_{n=p}^{q} a_n$ for $a_p + a_{p+1} + \cdots + a_q$.

We associate with (a_n) the seq of partial sums

$$S_n = \sum_{k=0}^{n} a_k$$

The expressions

$$a_0 + a_1 + a_2 + \cdots \quad \text{and} \quad \sum_{n \in \mathbb{N}} a_n$$

are called series and denote the value of $\lim_{n\to\infty} S_n$ if the limit exists

We say $\sum_{n \in \mathbb{N}} a_n$ converges/diverges if (S_n) converges/diverges.

Thm 3.11 (Cauchy criterion) becomes

Thm 3.22 $\sum a_n$ converges iff

$$\forall \varepsilon > 0 \ \exists N \ \forall n \geq m \geq N \quad \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

Pf: Follows from Thm 3.11 and fact that

$$\left| \sum_{k=m}^{n} a_k \right| = |S_n - S_{m-1}|$$

where S_n defined as above

$\square$

Taking $n=m$ above

Thm 3.23: If $\sum a_n$ converges then $a_n \to 0$

Obs: Converse is false. $\frac{1}{n} \to 0$ but $\sum_{n\in\mathbb{Z}_+} \frac{1}{n}$ diverges

Thm 3.14 becomes:

Thm 3.24: If $a_n \geq 0$ for all n then $\sum a_n$ converges if and only if its partial sums are bounded. $\square$

Thm 3.25 (Comparison Test)

① If $|a_n| \leq c_n$ for all $n \geq N$ and $\sum c_n$ converges then $\sum a_n$ converges

② If $a_n \geq d_n \geq 0$ for all $n \geq N$ and $\sum d_n$ diverges then $\sum a_n$ diverges

Pf: ① Given $\varepsilon > 0$ pick $M \geq N$ with $\forall\, m \geq n \geq M \quad \sum_{k=n}^{m} c_k < \varepsilon$.

Then for $m \geq n \geq M$

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon$$

Thus $\sum a_n$ converges by Thm 3.22.

② Follows from contrapositive of ①. $\square$

Thm 3.26: Let $z \in \mathbb{C}$. If $|z| < 1$ then $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$. If $|z| \geq 1$ then $\sum z^n$ diverges.

Note: $\sum_{n=0}^{\infty} z^n$ is called a Geometric Series

Pf: Assume $|z|<1$. Define $s_n = \sum_{k=0}^{n} z^k = 1 + z + z^2 + \cdots + z^n$.

Then $s_n = \frac{1 - z^{n+1}}{1-z}$ so $s_n \to \frac{1}{1-z}$ by Thm 3.3, 3.20 (E)

So $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

If $|z| \geq 1$ then $z^n \not\to 0$ so $\sum_{n=0}^{\infty} z^n$ diverges by Thm 3.23 $\square$

Thm 3.27: Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Longleftrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges

Pf: Set
$$s_n = a_1 + a_2 + \cdots + a_n$$
$$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$$
for all $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$. By Thm 3.24

$\sum a_n$ converges $\Leftrightarrow$ (s_n) bounded above

$\sum 2^k a_{2^k}$ converges $\Leftrightarrow$ (t_k) bounded above

($\Rightarrow$) Assume (s_n) bounded above by M.
Pick any $k \in \mathbb{N}$. Pick any $n > 2^k$. Then

$$2M \geq 2s_n \geq 2a_1 + 2a_2 + 2a_3 + 2a_4 + \cdots + 2a_{2^{k-1}+1} + \cdots + 2a_{2^k}$$

$$\geq a_1 + 2a_2 + 2(a_3 + a_4) + \cdots + 2(a_{2^{k-1}+1} + \cdots a_{2^k})$$

$$\geq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k$$

So (t_k) bounded above.

($\Leftarrow$) Assume (t_k) bounded above by M.
Pick any $n \in \mathbb{Z}_+$. Pick any k with $n < 2^k$. Then

$$s_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1})$$

$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k \leq M$$

So (s_n) bounded above. $\square$

Thm 3.28: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

Pf: When $p \leq 0$, $\sum \frac{1}{n^p}$ diverges since $\frac{1}{n^p} \not\to 0$ (Thm 3.23).
Assume $p > 0$. Then $\sum \frac{1}{n^p}$ converges iff

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \left(2^{(1-p)}\right)^k \text{ converges}$$

This last series is geometric, so converges iff $2^{1-p} < 1$ iff $p > 1$. $\square$

Thm 3.29: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$, diverges if $p \leq 1$.

Pf: Done if $p \leq 0$. Assume $p > 0$. By Thm 3.27 series converges iff $\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k (\log 2^k)^p}$

$$= \sum_{k=1}^{\infty} \frac{1}{k^p (\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges. Apply Thm 3.28 $\square$

Defn: $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828\ldots$ $\quad (0! = 1,\ n! = n(n-1)!)$

Obs: $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ so $\sum \frac{1}{n!}$ converges by comparison to geometric series $2 \cdot \sum \frac{1}{2^n}$.

Thm 3.31: $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$