The following are a set of questions that are similar to questions that will be on the Final.

Q1. a) Let $p$ be a permutation of $1, 2, \ldots, n$, and $P$ be the permutation matrix defined by $p$. That means that each entry of $P$ is either 0 or 1, with the property that for every row $i$, $1 \leq i \leq n$, $P(i, p(i)) = 1$ and $P(i, j) = 0$ for all $j \neq p(i)$. Show that $P^TP = I$.

b) Let $A$ be an $n \times n$, symmetric matrix and let $B$ be the result of $A$ after just one step of Gaussian elimination (so $B$ satisfies $b_{i1} = 0$ for all $i = 2, 3, \ldots, n$). Prove that $b_{ij} = b_{ji}$, for all $2 \leq i, j \leq n$. Be sure to label exactly where you use the symmetry of $A$.

Q2. Consider an iterative method for solving $Ax = b$ defined by a “splitting” $A = M - N$, and the iterating equation

$$Mx^{(k+1)} = Nx^{(k)} + b,$$

for $M = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, and $N = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Assuming that $x^*$ is the true solution for $Ax = b$, let $e^{(k)} = x^{(k)} - x^*$ for all $k$. Use the iterating equation $Mx^{(k+1)} = Nx^{(k)} + b$ to obtain a simpler iterating equation for $e^{(k)}$ (you will need the facts that $Ax^* = b$ and $A = M - N$).

Given $M, N$ above, will this iterative method converge for any initial condition $x^{(0)}$?

Q3. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 3 \end{bmatrix}$. Will the Gauss Seidel iteration converge regardless of the initial guess $x^{(0)}$? Explain your answer.

Q4. Write a MATLAB function that takes as input a matrix $A$, a vector $b$, an initial guess $x^{(0)}$, and a maximal number of iterations $k$, and then uses the Gauss Seidel method to solve $Ax = b$ with initial guess $x^{(0)}$. Your function should stop when the maximal number $k$ of iterations is reached.

Q5. Find the $PA = LU$ factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 2 & 4 \\ -2 & 4 & -1 \end{bmatrix}$ using Gauss elimination with partial pivoting. Make sure you write out the matrices $P, L, U$. Remember that you can always check your answer by computing $A = \text{PT}LU$.

Q6. To answer the questions below, you may assume that the following holds (you don’t need to prove this):

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$
a) Show that \( \|A\|_1 \leq n \|A\|_\infty \).

b) Find an example of a matrix for \( n = 2 \) such that \( \|A\|_1 = 2 \|A\|_\infty \). You must compute both norms to show that your example works.

Hint: It may help to first find an example of a length 2 vector \( x \) for which \( \|x\|_1 = 2 \|x\|_\infty \) first.

c) Let \( Q \) be an orthogonal matrix. Show that \( \|Qx\|_2 = \|x\|_2 \) for every vector \( x \).

Q7. Let \( u \) be a unit-norm vector (\( \|u\|_2 = 1 \)) and let \( Q_u \) be the Householder reflector \( Q_u = I - 2uu^T \). What are the eigenvalues and determinant of \( Q_u \)? (Hint: recall that \( P = uu^T \) is a rank-one matrix for which \( Pu = u \).)

Q8. Consider the positive definite matrix \( A \in \mathbb{R}^{n \times n} \), and let \( A = R^T R \) be its Cholesky decomposition.

a) Show that \( \kappa_2(A) = (\kappa_2(R))^2 \), where \( \kappa_2 \) stands for “condition number in norm 2”.

b) Show that the right singular vectors of \( A \) are also right singular vectors for \( R \).

Q9. Suppose that the matrix \( A \in \mathbb{C}^{4 \times 4} \) has eigenvalues \( \{2, -2, 1.5, 0.5\} \), with eigenvectors \( v_1, v_2, v_3, v_4 \). Let \( q = v_3 + 2v_4 \). Will the power method started with \( q_0 = q \) converge, and if so, to what? (Hint: without normalizing the vectors, try writing out \( q_1, q_2, q_3, q_4, \ldots \))

Q10. Let

\[
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

a) Perform the Gram-Schmidt process on \( \{v_1, v_2, v_3\} \) to obtain three orthonormal vectors \( q_1, q_2, q_3 \).

b) Let \( A = [v_1, v_2, v_3] \); use a) to find the minimizer \( x \) for the least squares problem involving \( A \) and \( b = [1, 1, 1, 1]^T \).

Q11. The following is an excerpt from a MATLAB code:

\[
[U, S, V] = svd(A);
S = diag(S);
r = rank(A);
S1 = S(1:r,1:r);
S2 = diag(ones(r,1)./S(1:r));
X = V(:,1:r)*S2*U(:,1:r)';
Y = U(:,1:r)*S1*V(:,1:r)';
\]

Read and understand the code, then answer the following questions:
a) What is $X$?

b) In exact arithmetic, what is $\text{norm}(A-Y)$?

Q12. Let $A \in \mathbb{C}^{2 \times 2}$ be a defective matrix. Show that $A$ is similar to a matrix

$$B = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix},$$

for some particular choices of $\lambda$ and $\alpha \neq 0$ in $\mathbb{C}^2$. 