Chapter 1.4: The Cholesky decomposition/factorization

A Special Case: Cholesky Decomposition. We say that a matrix $A$ is positive definite if it is symmetric and, for any $x$ an $n \times 1$ real vector with $x \neq 0$, $x^T Ax > 0$.

A simple consequence of the positive definiteness condition is the fact that a positive definite $A$ is non-singular. Indeed, if this were not the case, there would be a $y \neq 0$ such that $Ay = 0$, and in this case we would have $y^T Ay = 0$. Since the matrix is positive definite, this cannot happen, and thus such an $y$ cannot exist and so $A$ must be non-singular.

In the last part of the first video (Lecture 6, part 1) I give a full proof for the theorem below; but as I will not require it for the class, I will give here the theorem without proof.

**Theorem 1.** If $A$ is positive definite, there exists an upper triangular matrix $R$ such that

$$A = R^T R.$$

This decomposition is unique, and it is called the Cholesky Decomposition.

One of the proofs of the theorem (given in the Lecture 6, part 2 video) is based on the fact that a positive definite matrix $A$ has an $LU$ (and thus, an $LDV$ decomposition). But, while one could obtain $R$ from $A$ via the $LU$ factorization, it is more advantageous to use the symmetry of the problem, and solve the system directly.

We do this by noting that if we write

$$R^T R = \begin{bmatrix} r_{11} & 0 & \ldots & 0 \\ r_{12} & r_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \ldots & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ 0 & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{12} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix} = A,$$

then multiplying $R^T R$ we can obtain

$$r_{11}^2 = a_{11},$$

$$r_{11}r_{1j} = a_{1j}, \quad \text{for all } j \geq 1,$$

and this gives formulas for all the entries in the first row of $r$.

Going further, we note that for any $i$,

$$r_{11}^2 + r_{2i}^2 + \ldots + r_{ii}^2 = a_{ii},$$

$$r_{1i}r_{1j} + r_{2i}r_{2j} + \ldots + r_{(i-1)j}r_{(i-1)j} + r_{ij}r_{ij} = a_{ij}, \quad \text{for all } j \geq i + 1,$$

we see that we can solve recurrently for all entries of $R$, via the following code.

```matlab
function R = cholesky_factor(A);
    n = size(A,1);
```
\[
R = \text{triu}(A,0); \quad \% \text{we initialize } R \text{ as the upper triangular part of } A
\]

\text{for } i = 1 : n
\begin{align*}
& \text{for } k = 1 : (i - 1) \\
& \quad R(i,i) = R(i,i) - R(k,i)^2; \\
& \quad \text{end}
\end{align*}

\text{if } R(i,i) \leq 0, \text{ error('A is not positive definite')}, \text{ end}
\begin{align*}
R(i,i) &= \text{sqrt}(R(i,i)); \\
\text{for } j = (i + 1) : n \\
& \quad \text{for } k = 1 : (i - 1) \\
& \quad \quad R(i,j) = R(i,j) - R(k,i) \times R(k,j); \\
& \quad \quad \text{end}
& \quad R(i,j) = R(i,j)/R(i,i); \\
& \quad \text{end}
\end{align*}
\text{end}

If we think of \textit{sqrt} as one operation, a quick calculation shows a flop count of
\[
\sum_{i=1}^{n} ((2i - 2) + 1 + (n - i) \cdot (2(i - 1) + 1)) = n^3 - 2n^3/3 + O(n^2) = n^3/3 + O(n^2); 
\]
you will notice this is half the cost of \textit{LU} factorization.