Chapter 4.2: the SVD and the spectral norm and condition number

The spectral norm and the SVD. Recall the singular value decomposition, \( A = U \Sigma V^T \).

**Theorem 1.** For any matrix \( A \in \mathbb{R}^{n \times m} \), define the induced 2-norm \( \| A \|_2 = \max_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} \). (We have previously defined this solely for \( m = n \), but there is nothing to prevent us from extending this definition to any \( m, n \).)

Then \( \| A \|_2 = \sigma_1 \), where \( \sigma_1 \) is the largest singular value of \( A \).

**Proof.** Recall that \( \{ v_1, \ldots, v_m \} \) is an orthonormal basis for \( \mathbb{R}^m \), and \( \{ u_1, \ldots, u_n \} \) is an orthonormal basis for \( \mathbb{R}^n \). Let \( x \in \mathbb{R}^m \), \( x \neq 0 \), then there exist constants \( c_i \) with \( 1 \leq i \leq m \) such that \( x = \sum_{i=1}^{m} c_i v_i \).

Note that since \( A = U \Sigma V^T \), \( AV = U \Sigma \), and \( Av_i = \sigma_i u_i \) for all \( 1 \leq i \leq m \).

Calculating, we obtain that

\[
\| x \|_2^2 = \langle \sum_{i=1}^{m} c_i v_i, \sum_{i=1}^{m} c_i v_i \rangle = \sum_{i,j=1}^{m} c_i c_j v_i^T v_j ,
\]

and due to the orthonormality of the set \( \{ v_1, \ldots, v_m \} \),

\[
\| x \|_2^2 = \sum_{i=1}^{m} c_i^2 , \quad \text{or} \quad \| x \|_2 = \sqrt{\sum_{i=1}^{m} c_i^2} .
\]

Similarly, we show that

\[
Ax = A \left( \sum_{i=1}^{n} c_i v_i \right) = \sum_{i=1}^{m} c_i Av_i = \sum_{i=1}^{m} c_i \sigma_i u_i ,
\]

and \( \| Ax \|_2 = \sqrt{\sum_{i=1}^{m} c_i^2 \sigma_i^2} .\)

But then

\[
\frac{\| Ax \|_2}{\| x \|_2} = \sqrt{\frac{\sum_{i=1}^{m} c_i^2 \sigma_i^2}{\sum_{i=1}^{m} c_i^2}} \leq \sqrt{\frac{\sum_{i=1}^{m} c_i^2 \sigma_1^2}{\sum_{i=1}^{m} c_i^2}} = \sigma_1 ,
\]

where the inequality in the above is based on the fact that \( \sigma_1 \) is the largest singular value. Thus, by taking the maximum, we obtain also that

\[
\| A \|_2 = \max_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} \leq \sigma_1 .
\]

On the other hand, by making \( x = v_1 \), and hence \( Ax = Av_1 = \sigma_1 u_1 \), we obtain that

\[
\sigma_1 = \frac{\| Av_1 \|_2}{\| v_1 \|_2} \leq \max_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} = \| A \|_2 .
\]

These two inequalities taken together show the desired conclusion. \( \square \)
We thus have an expression of the induced 2-norm (also known as the spectral norm) for any matrix, as the largest singular value of the matrix.

The SVD of the inverse and condition numbers. For an $n \times n$ square, invertible matrix $(r = n)$, is $A = U \Sigma V^T$ (then $U, V, \Sigma$ are all $n \times n$), $A^{-1} = V \Sigma^{-1} U^T$, and note that this means the singular values of $A^{-1}$ are $\frac{1}{\sigma_n} \geq \frac{1}{\sigma_{n-1}} \geq \ldots \geq \frac{1}{\sigma_1}$, and thus
\[
||A^{-1}||_2 = \frac{1}{\sigma_n}.
\]

Thus,

**Theorem 2.** For an $n \times n$ invertible matrix $A$, in the 2-norm,
\[
\kappa(A) = ||A||_2 \cdot ||A^{-1}||_2 = \frac{\sigma_1}{\sigma_n}.
\]

Low-rank approximation. Let us write $A$ as the sum of rank-one matrices (as we have seen in the previous lecture),
\[
A = \sum_{i=1}^{r} \sigma_i u_i v_i^T.
\]

We will denote by $A_k$ the truncated sum $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$.

**Theorem 3.** For any $k = 1, \ldots, r$, let $V_k$ be the set of all $n \times m$ matrices of rank at most $k$. Then, with the notation above,
\[
\sigma_{k+1} = ||A - A_k||_2 = \min_{B \in V_k} ||A - B||_2.
\]

That is, among all $n \times m$ matrices of rank $k$ or less, $A_k$ is “closest” to $A$, and the difference in norm is $\sigma_{k+1}$.

**Remark 1.** Note that when $m = n$, $\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{R}^{n \times n}$, and that $V_{n-1}$ is the set of all singular matrices (if a matrix is singular, then its rank is at most $n - 1$.)

The proof for this theorem is in the textbook (4.2.15), and I invite you all to take a look. Although it’s a bit long, it is quite elegant.

As a consequence, we have the following two results.

**Corollary 1.** If $A$ is square and full-rank ($m = r = n$) and $B$ is a matrix for which $||B - A||_2 < \sigma_n$, then $B$ is full-rank.

**Corollary 2.** If $A$ is square and non-singular ($m = n = r$ and $\sigma_n > 0$), let $B$ be the matrix that is singular and closest to $A$ in 2-norm (so that $||A - B||_2$ is minimal among all singular matrices). Then, with the notations above, $B = A_{n-1}$ and
\[
\frac{||A - A_{n-1}||_2}{||A||_2} = \frac{1}{\kappa(A)}.
\]

Note that the second corollary shows that the result we got in 2.3 about the distance to singularity and the condition number is, in fact, tight.