Chapter 5.1+: Motivation and applications for eigenvalues and eigenvectors

Section 5.1. There are myriad applications in physics and engineering that involve finding the eigenvalues and eigenvectors of a given matrix. Section 5.1 of the book lists a few of them—electrical networks, spring dynamics, stability of linear systems. All of these examples contain ODE (ordinary differential equations) systems of the type $\dot{x} = Ax - b$, where $x = (x_1, x_2, \ldots, x_n)^T$ is a vector of functions, generally depending on a time variable $t$, and $\dot{x} = \frac{dx}{dt}$ is the vector of derivatives with respect to $t$.

One important step in solving such ODEs is to solve the homogeneous part of the system first, that is, deal with the case when $b = 0$. In this case, one needs to solve the system

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \quad \forall \ 1 \leq i \leq n.$$  

The common technique to solve this system is to look for a solution where all the $x_i$s are constant multiples of the same scalar function $g(t)$, that is, we assume a solution of the form $x(t) = g(t)v$, with $v$ a vector and $g(t)$ a scalar function of $t$.

For such a solution, we see that the equation becomes

$$\dot{g}(t)v = g(t)Av,$$

or equivalently $\frac{\dot{g}(t)}{g(t)}v = Av$. As $A$ and $v$ are constants, this means that there exists some constant $\lambda$ such that $\frac{\dot{g}(t)}{g(t)} = \lambda$, and also that $Av = \lambda v$. So $(\lambda, v)$ is an eigenpair of $A$. Note that as $\frac{\dot{g}(t)}{g(t)} = (\ln g(t))'$, this implies that there exists some constant $c$ for which $g(t) = ce^{\lambda t}$, and thus that $x(t) = ce^{\lambda t}v$.

Assume now that the matrix $A$ has a full complement (that is, $n$) of linearly independent eigenvectors $v_1, \ldots, v_n$, with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Then any linear combination of $e^{\lambda_i t}v_i$,

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}v_i$$  \hspace{1cm} (1)

is a solution to the homogeneous system $\dot{x} = Ax$. In other words, the space of solutions with this form has dimension $n$, as $v_1, \ldots, v_n$ are linearly independent.

Basic ODE theory guarantees that if we fix the initial conditions $x(0) = x_0$, with $x_0 \neq 0$, the solution with this boundary conditions exists and is unique up to any time $t$. This implies that all solutions belong to a $n$-dimensional space, as $x_0$ is $n$-dimensional. Hence this space is precisely the one described by (1).

Side note: If the matrix $A$ does not have a full complement of eigenvectors (in other words, it is defective), then the space of solutions will be a bit different, but the answer can still be obtained.

Armed with the solution for the homogeneous equation, we can solve the inhomogeneous one as follows. Let $x_*$ be a solution to $Ax = b$; clearly $x_*$ is a solution to the inhomogeneous one since in
this case $\dot{x}_* = 0$. But then $z = x_* + x$, where $x$ is given by (1), is a solution to the inhomogeneous system; moreover, given any two distinct solutions to the inhomogeneous system, their difference is a solution to the homogeneous one.

Indeed, if $y_1$ and $y_2$ both solve $\dot{x} = Ax - b$, then by subtracting the two systems we obtain that

$$\begin{align*}
(y_1 - y_2) &= A(y_1 - y_2),
\end{align*}$$

so $y_1 - y_2$ is a solution of the homogeneous system and has the form (1).

But then it follows that any solution to the inhomogeneous system is of the form $x_* + x$, where $x$ is given by (1). So, eigenvalues and eigenvectors of $A$ are crucial in constructing solutions to ODEs with constant coefficients.

(They are also crucial in constructing solutions to systems with variable coefficients, but this goes beyond the scope of Math 170A.)

**Principal Component Analysis (PCA).** A second motivation for studying eigenvalues and eigenvectors comes from one of the fundamental problems in statistics (and, by extension, in data science). When dealing with reams of data, there are a few important things to keep in mind in order to separate the chaff; one of them is redundancy, that is, how much of your data is dependent on itself. The second one has to deal with finding the best prism to see the data through; which information contained in the data is important, and to what degree?

Here is where PCA comes in, and I will illustrate it on an example.

Suppose we have a sample of $m$ cells, some of which may be of different type (e.g., cancerous); on the surface all look the same, but one can measure the expression of certain genes in each cell, and one would expect that different cells have different ways of expressing those genes. Suppose we perform $n$ tests, on $n$ genes, throughout the population of cells, and record the results in a table:

<table>
<thead>
<tr>
<th>Gene 1</th>
<th>Cell 1</th>
<th>Cell 2</th>
<th>...</th>
<th>Cell m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gene 2</td>
<td>$x_{11}$</td>
<td>$x_{12}$</td>
<td>...</td>
<td>$x_{1m}$</td>
</tr>
<tr>
<td>Gene 2</td>
<td>$x_{21}$</td>
<td>$x_{22}$</td>
<td>...</td>
<td>$x_{2m}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Gene n</td>
<td>$x_{n1}$</td>
<td>$x_{n2}$</td>
<td>...</td>
<td>$x_{nm}$</td>
</tr>
</tbody>
</table>

This clearly gives us a matrix $X \in \mathbb{R}^{n \times m}$. Suppose now that the columns of $x$ have been centered, so that the mean of each column is 0; this will make it easier to talk about independence, and it also has the benefit of getting rid of the potential bias of the cell.

We will examine the sample covariance matrix of $X$, which is the matrix $C = X^T X$. By looking at the entry $c_{ij}$ in $C$, we see that it corresponds to the dot product of column $i$ with column $j$; if the cells $i$ and $j$ are of the same type, we expect that column $i$ and column $j$ of $X$ are very close to each other, and the inner product will be close to the square of their norm, so large; whereas we expect that if cells $i$ and $j$ are of different types, the fluctuations from the mean represented by the columns will be independent, and so we expect that $c_{ij}$ is close to 0, so small.

On the other hand, $C = X^T X$ is symmetric, and so it will have an eigenvalue decomposition $C = QAQ^T$, with $Q$ orthogonal and $A$ diagonal with the eigenvalues of $C$ on the diagonal (the eigenvectors corresponding to these eigenvalues are the columns of $Q$).

Recall that the rank of $C$ will be the number of nonzero eigenvalues of $C$. At the same time, the number of independent columns of $X$. It will be an exercise for you later to show that the rank of $X$ is the same as the rank of $C$, but for now let’s take it as a given.
Then, the number of nonzero eigenvalues of \( C \) tells us how many independent columns of \( X \) we have. Given that the columns corresponding to the same type of cell are dependent, the rank has the benefit of telling us how many different types of cells there are!

In fact, the eigenvalue decomposition of \( C \) tells us all we need to know: how many eigenvectors (principal components) correspond to nonzero eigenvalues. The eigenvector corresponding to the largest eigenvalue is the first principal component, the one corresponding to the second largest eigenvector is the second principal component, etc.

We now examine the matrix \( W = XQ \), which simply transforms the data to a new coordinate system; the covariance of \( W \) is

\[
W^TW = (XQ)^T XQ = QTXXQ = QTCCQ = QT(Q\Lambda Q^T)Q = \Lambda ,
\]

which is a diagonal matrix. The new data columns \([w_1,\ldots,w_m]\) are independent! So, in addition to helping us identify and eliminate redundancies, expressing the data as \( W \) rather than \( X \) provides a better understanding of the information contained within, since we have rendered the variables independent and also identified which gene expressions are the most important and to what degree.

There are other considerations (e.g., having to do with the noise contained in the samples) that make this problem a bit more complex than the scenario outlined here, but the bottom line is that, once again, eigenvalues and eigenvectors help us to get the root of an important problem which has very wide applicability.