Homework problems that will be graded (P1 - P6, 30pts in total):

**P1.** (This is similar to exercise 4.2.10 from the textbook)
Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$.

a) Use the SVD of $A$ to deduce the SVD of $A^T A$.

b) If $m = n$ and $A$ is full-rank, use a) to show that $\|A^T A\|_2 = \|A\|_2^2$ and $\kappa_2(A^T A) = \kappa_2(A)^2$.

**P2.** Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, $\text{rank}(A) = n$. Determine the singular value decomposition of $(A^T A)^{-1}$, $(A^T A)^{-1} A^T$, $A(A^T A)^{-1}$, $A(A^T A)^{-1} A^T$ in terms of the SVD of $A = U \Sigma V^T$. Show that the second matrix above is $A^\dagger$.

**P3.** (This is similar to exercise 4.3.9 from the textbook)
Work this exercise using pencil and paper. You can use MATLAB to check your work.
Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$ 

a) Find the rank $r$ of $A$.

b) Deduce what the reduced SVD of $A$ should look like. Specifically, what are the sizes of $U_r$, $\Sigma_r$, and $V_r$?

c) Knowing that the columns of $U_r$ and $V_r$ are orthonormal, figure out $U_r$, $V_r$, and $\Sigma_r$.

d) Calculate $A^\dagger$.

e) Calculate the minimum norm solution of the least-squares problem for the over-determined system $Ax = b$.

**P4.** Let $A \in \mathbb{R}^{n \times n}$, invertible. In Homework 4, you have found that

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ are the singular values of $A$. Find $\kappa_F(A)$, the condition number for $A$ in Frobenius norm, in terms of the singular values of $A$. (You do not need to reprove the formula for $\|A\|_F$ above.)
P5. (This is similar to exercise 5.1.24 from the textbook)

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the $2 \times 2$ identity matrix.

(a) Show that the characteristic equation of $A$ is
\[ \lambda^2 - 2\lambda + 1 = 0. \]

Its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1$.

(b) We now perturb one coefficient of the characteristic polynomial slightly and consider the equation
\[ \lambda^2 - 2\lambda + (1 - \varepsilon) = 0, \]

where $0 < \varepsilon \ll 1$. Solve the equation for the roots $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

(c) Show that when $\varepsilon = 10^{-12}$, $|\hat{\lambda}_1 - \lambda_1|$ and $|\hat{\lambda}_2 - \lambda_2|$ are one million times bigger than $\varepsilon$.

(d) Sketch the graphs of the original and perturbed polynomials (using some $\varepsilon$ bigger than $10^{-12}$), to give some indication why the roots are so sensitive to the $\varepsilon$ perturbation.

P6. (MATLAB problem)

Let
\[
A = \begin{bmatrix}
1 & 2 & 4 \\
2 & 0 & 4 \\
1 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix}.
\]

a) Use the MATLAB function rank to figure the rank of $A$.

b) MATLAB’s pinv command returns the pseudoinverse of a matrix. Use pinv to compute the pseudoinverse $A^\dagger$ of $A$.

c) Use MATLAB’s svd command to find the singular values of $AA^\dagger$ and $A^\dagger A$. Explain why these results are what you would expect. What are $AA^\dagger$ and $A^\dagger A$?

Additional problems, which will not be graded (A1 - A3):

A1. (Similar to exercise 4.2.25 of the textbook)

a) Let $A \in \mathbb{R}^{2 \times 2}$ with singular values $\sigma_1 \geq \sigma_2 > 0$. Show that the set $\{Ax : \|x\|_2 = 1\}$ (image of a unit circle) is an ellipse in $\mathbb{R}^{2 \times 2}$ whose major and minor semiaxes have lengths $\sigma_1$ and $\sigma_2$, respectively.
b) Let $A \in \mathbb{R}^{m \times n}, m \geq n, \text{rank}(A) = n$, with singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. Show that the set $\{Ax : \|x\|_2 = 1\}$ (image of a unit circle) is an $n$-dimensional hyperellipsoid with semiaxes $\sigma_1, \sigma_2, \ldots, \sigma_n$.

**A2.** (Similar to Exercise 4.3.14 from the textbook.)

Let $A \in \mathbb{R}^{m \times n}$ and let $A^\dagger$ be its pseudoinverse. Show that the following relationships hold:

\[
A^\dagger A A^\dagger = A^\dagger \quad (A^\dagger A)^T = A^\dagger A \\
A A^\dagger A = A \quad (A A^\dagger)^T = A A^\dagger
\]

**A3.** Consider the positive definite matrix $A \in \mathbb{R}^{n \times n}$, and let $A = R^T R$ be its Cholesky decomposition. Show that $\kappa_2(A) = (\kappa_2(R))^2$, where $\kappa_2$ stands for “condition number in norm 2”.