Homework problems that will be graded (P1 - P6, 30pts in total):

**P1.** (This is similar to exercise 5.2.17 from the textbook)
Let \( A \in \mathbb{R}^{m \times n} \) have SVD \( A = U \Sigma V^T \). Show that the columns \( v_1, \ldots, v_n \) of \( V \) are linearly independent eigenvectors of \( A^T A \) and that the columns \( u_1, \ldots, u_m \) of \( U \) are linearly independent eigenvectors of \( AA^T \), corresponding to the eigenvalues \( \sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0 \), where \( r \) is the rank of the matrix \( A \).

**P2.** Work the following problem by hand. Find the SVD of
\[
A = \begin{bmatrix}
3 & 2 & 2 \\
2 & 3 & -2
\end{bmatrix}.
\]

*Hint:* Use P1. Remember that you can check your result with MATLAB’s svd command.

**P3.** (This is similar to exercise 5.2.20 from the textbook)
Let \( A \in \mathbb{C}^{n \times n} \) be a block-triangular matrix
\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix},
\]
where \( A_{11} \in \mathbb{C}^{j \times j} \) and \( A_{22} \in \mathbb{C}^{k \times k} \), \( j + k = n \).

a) Let \( \lambda \) be an eigenvalue of \( A_{11} \) with eigenvector \( v \). Show that \([v,0]^T\) is an eigenvector of \( A \) with eigenvalue \( \lambda \), where the 0 here is a vector of 0s of length \( k \).

b) Suppose now that \( \lambda \) is not an eigenvalue of \( A_{11} \), but is an eigenvalue of \( A \). Let \([v, w]^T\) be the associated eigenvector for \( A \), with \( w \) of length \( k \) and \( v \) of length \( j \). Show that \( w \) is an eigenvector of \( A_{22} \) with eigenvalue \( \lambda \).

c) Let \( \lambda \) be an eigenvalue of \( A \) with eigenvector \([v, w]^T\), where \( v \) has length \( j \) and \( w \) has length \( k \). Show that either \( v \) is an eigenvector of \( A_{11} \), or \( w \) is an eigenvector of \( A_{22} \) (Hint: consider the cases \( w = 0 \) and \( w \neq 0 \)).

d) Using a), b), c), explain what an eigenpair \((\lambda, v)\) of \( A \) looks like.

**P4.** (This is similar to exercise 5.3.8 from the textbook)
Work this problem by hand. Let
\[
A = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
Carry out the power method with starting vector \( q_0 = [a, b]^T \), where \( a, b \geq 0 \) and \( a \neq b \). Explain why the sequence fails to converge. What is the problem with the convergence argument we had in the lecture?
P5. (This is similar to exercise 5.4.47 from the textbook)

A matrix \( A \in \mathbb{C}^{n \times n} \) is called \textit{normal} if \( AA^* = A^*A \). Here \( A^* \) denotes the complex conjugate transpose of \( A \): \( A^* = \overline{A^T} \).

Suppose \( A \) is normal, with eigenvalues \( \lambda_1, \ldots, \lambda_n \) ordered by \( |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \) and singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0 \).

Show that \( \sigma_j = |\lambda_j| \) for all \( 1 \leq j \leq n \). What are the eigenvectors of \( A \)?

P6. (MATLAB problem) Run the sni.m code provided. The first 4 lines construct a random matrix \( B \) with prescribed eigenvalues. The fifth line inverts \( B \) and names the inverse \( C \), and the remainder of the code applies the power method to the matrix \( C \). Run the code and examine it closely, then answer the following questions.

a) What are the eigenvalues of \( B \) and \( C \)? Your answer should be in terms of the eigenvalues of \( A \).

b) Use reasoning, not MATLAB, to answer this question, and show your reasoning.

\( q \) is a very good approximation for an eigenvector of \( A \). What is the corresponding eigenvalue \( \lambda \)?

c) How would the answer in b) change if we replaced 0.25 with 1.5 in line 4 of the script? Explain your answer.

d) Show how good the approximation vector \( q \) is by typing in MATLAB

\[
\text{norm}((A-\lambda*\text{eye}(5))*q),
\]

where \( \lambda \) is the number you obtained in b). This computes the 2-norm of the vector in parentheses; if it is small, the vector \( q \) is a good approximation.