Section 1.5: In practical applications, one often encounters cases in which variables are related locally (as is the case with the mass-spring problem, or really any problem resulting from the discretization of an ODE). This leads to banded matrices, i.e., matrices with just a few non-zero diagonals (see example below for a matrix with 2 lower diagonals, in orange and black, and one upper one in blue, in addition to the main one in red).

\[
\begin{bmatrix}
  a_{11} & a_{12} & 0 & 0 & 0 \\
  a_{21} & a_{22} & a_{23} & 0 & 0 \\
  a_{31} & a_{32} & a_{33} & a_{34} & 0 \\
  0 & a_{42} & a_{43} & a_{44} & a_{45} \\
  0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\
  0 & 0 & 0 & a_{64} & a_{65} & a_{66} \\
\end{bmatrix}
\]

For such matrices with \(s\) lower diagonals and \(t\) upper diagonals in addition to the main one, if the \(LU\) factorization exists, the default algorithm/code for computing it is inefficient. Indeed, the algorithm does a lot of work to zero-out entries which are already 0 (and computing multipliers that are exactly 0). In fact, it should not be particularly hard to see from the algorithm that if the factorization exists, \(L\) should be lower triangular with \(s\) lower diagonals (in addition to the main one, which is all ones), and \(U\) should be upper triangular with \(t\) upper diagonals in addition to the main one.

To be clear, all the algorithm should do in order to become efficient is to restrict the range of \(j\) and \(k\) to accommodate the bandwidth. Through this restriction we do significantly less work (\(O(n)\) flops, as opposed to \(O(n^3)\)).

Similarly, if \(A\) is positive definite and banded with a total of \(2s + 1\) diagonals (sometimes called the band), then \(R\) will have the same number of “extra” diagonals as \(A\) (a total of \(s + 1\), main one plus the upper ones). Sometimes \(2s + 1\) is called the “band”, while \(s\) is called the “semiband”. The Cholesky factorization algorithm can be modified accordingly to be efficient.

Wrapping up. Since in this chapter we are mainly interested in solving linear equations, and since for those Gaussian Elimination computes only implicitly \(L\) and \(U\), what is it that the \(LU\) factorization (or \(R^T R\) Cholesky decomposition) bring to the table?

The answer is two-fold. One reason why they are interesting is the set of theoretical properties they encode, and the fact that they can be used to prove and understand a lot of theoretical results about \(A\). The second, and more weighty, given the focus of this class, is in applications. Often, one has to solve a whole set of linear systems using the same matrix \(A\) and various right hand sides \(b_k\); these requests may arrive at the same time or sequentially. Especially in the latter case, it makes sense to keep on hand the \(LU\), \(P^T LU\), or \(R^T R\) decomposition of \(A\) (whichever one is appropriate), so that as the new \(b_k\) arrive, the solution can be quickly constructed via a forward substitution, followed by a backward one. This way, each additional request can be completed with \(O(n^2)\) work, rather than the usual \(O(n^3)\) (and the \(O(n^3)\) work is only performed once, in finding the factorization.)
2.1 Norms. If up to this point we have talked about exactly solving systems of equation, in the next module we will start thinking about approximation, or finding best fits, or sometimes projections.

The first step toward this is defining distances, so we can measure how good the approximation is (or, in certain cases, to ensure we can prove that iterative methods converge, or to understand how the matrix properties may affect the size of the floating point error).

The simplest distance between two vectors is, naturally, the Euclidean one:

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \ldots + |x_n - y_n|^2}.$$ 

What makes this distance very natural is that is given by a norm (which we define below); it is a handy way to measure distance, but it isn’t always the best, and we can define many, many more. The most important distances are given by norms.

**Definition** A norm is a function $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ which satisfies the following three properties:

- $||x|| > 0$ for all $x \neq 0$; (positivity)
- $||\alpha x|| = |\alpha| \cdot ||x||$, for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$, (multiplication by a constant)
- $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in \mathbb{R}^n$. (triangle inequality)

**Remark 1.** Any norm defines a distance via $d(x, y) = ||x - y||$.

**Examples of norms:**

- $||x||_1 = |x_1| + |x_2| + \ldots + |x_n|$, 
- $||x||_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$, 
- $||x||_p = \left( |x_1|^p + |x_2|^p + \ldots + |x_n|^p \right)^{1/p}$,  

for any $p \geq 1$ a real number. The above are called “$p$”-norms; the 1-norm is sometimes known as the “Manhattan norm”, the 2-norm is the Euclidean norm.

Also if $|| \cdot ||$ is a norm and $A$ is a matrix, under certain circumstances, $||x||_A := ||Ax||$ will be a norm.

One interesting feature of these norms is the shape of the unit sphere (the set of points of norm 1). For example, for $n = 2$, the unit sphere in 1-norm is the square whose vertices are at $(\pm 1, 0)$ and $(0, \pm 1)$. In the 2-norm, the unit sphere is simple the unit circle. As we take a larger and larger $p$-norm, the unit sphere flattens, and approaches the square with vertices at $(\pm 1, \pm 1)$. This is not accidental, but rather due to the existence of the $\infty$-norm:  

$$||x||_\infty = \max_i |x_i|,$$

for which the above mentioned square is the unit sphere.

**Matrix norms.** All matrices can be seen as vectors with extra structure. We will focus mostly on square, $n \times n$ matrices, although one can define matrix (semi)norms for rectangular matrices as well.

**Definition** A matrix norm is a function defined $|| \cdot || : \mathbb{R}^{n \times n} \to \mathbb{R}$ which satisfies the following four properties:
• $\|A\| > 0$ for all $A \neq 0$; (positivity)

• $\|\alpha A\| = |\alpha| \cdot \|A\|$, for all $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}^{n \times n}$, (multiplication by a constant)

• $\|A + B\| \leq \|A\| + \|B\|$, for all $A, B \in \mathbb{R}^{n \times n}$, (triangle inequality)

• $\|AB\| \leq \|A\| \cdot \|B\|$, for all $A, B \in \mathbb{R}^{n \times n}$. (submultiplicativity)

**Remark 2.** Note that the first three conditions are the same as for vectors, but the fourth (submultiplicativity) is new.

More next time.