Section 2.1: Norms. We started last time to talk about the special requirement that a matrix norm must satisfy: submultiplicativity. This is non-trivial. For example, examine the following matrix norm, also known as the Frobenius norm:

\[ ||A||_F = \sqrt{\sum_{i,j} |a_{ij}|^2} \]

This, at first glance, looks like the 2-norm for vectors. Indeed, the first three properties of this matrix norm follow straightforwardly from the fact that they are true for a vector norm. However, the fourth one is not immediate. Let us examine it:

\[ ||AB||_F^2 = \sum_{i,j} |AB_{ij}|^2 = \sum_{i,j} \left( \sum_{k=1}^n |a_{ik}b_{kj}| \right)^2 \]

(1)

The Cauchy-Schwarz theorem says that

\[ \left( \sum_{i=1}^n a_ib_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) ; \]

if we apply this to (1), we obtain that

\[ ||AB||_F^2 \leq \sum_{i,j} \left( \sum_{k=1}^n |a_{ik}| \right)^2 \left( \sum_{k=1}^n |b_{kj}|^2 \right) , \]

and so we may split the right hand side as

\[ \sum_{i,j} \left( \sum_{k=1}^n |a_{ik}| \right)^2 \left( \sum_{k=1}^n |b_{kj}|^2 \right) = \left( \sum_{i,k} |a_{ik}|^2 \right) \left( \sum_{k,j} |b_{kj}|^2 \right) = ||A||_F^2 \cdot ||B||_F^2 . \]

This proves submultiplicativity.

Induced matrix norms. Any vector norm induces a matrix one, as follows.

Definition Given the n-dimensional vector norm \( ||\cdot|| \), the induced matrix norm on \( n \times n \) matrices is defined as

\[ ||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||} . \]
The first three properties of the induced norm follow fairly easily from the three properties of the corresponding vector norm. The fourth is more subtle:

\[ ||AB|| = \max_{x \neq 0} \frac{||ABx||}{||x||} = \max_{x \neq 0, Bx \neq 0} \frac{||ABx||}{||Bx||} \cdot \frac{||Bx||}{||x||} \leq \max_{x \neq 0, Bx \neq 0} \frac{||ABx||}{||Bx||} \cdot \max_{x \neq 0} \frac{||Bx||}{||x||}. \]

Note that, by denoting \( y = Bx \), we can rewrite

\[ \max_{x \neq 0, Bx \neq 0} \frac{||ABx||}{||Bx||} = \max_{y \neq 0} \frac{||Ay||}{||y||} = ||A||, \]

and thus

\[ ||AB|| \leq ||A|| \max_{x \neq 0} \frac{||Bx||}{||x||} = ||A|| \cdot ||B||. \]

**Important matrix norms.** \( ||A||_1 \), the induced matrix 1-norm, can be seen to be

\[ ||A||_1 = \max_j \sum_{i=1}^n |a_{ij}|; \]

this is the maximum column 1-norm; \( ||A||_\infty \), the induced \( \infty \)-norm, is actually

\[ ||A||_\infty = \max_i \sum_{j=1}^n |a_{ij}|, \]

this is the maximum row 1-norm (NOT the maximum row \( \infty \)-norm!) Also, \( ||A||_2 \) and \( ||A||_F \) are different, as you can see from the fact that if \( I \) is the \( n \)-dimensional identity, \( ||I||_2 = 1 \), whereas \( ||I||_F = \sqrt{n} \).

**Section 3.1: Least Squares.** With this, we jump into the whole idea of approximation by discussing best fit. Given a collection of points in 2 dimensions, coming from some noisy measurements, we would like to find a curve that is the “best fit” (an idea that is perhaps a little nebulous) for the set of points.

The simplest case is when the curve in question is a line; given the points \((t_1, y_1), \ldots, (t_n, y_n)\), we would like to find the straight line passing through the set of points that stays closest to them. This is, in a way, a slightly ill-defined notion, and so we must make it clearer. A line is defined by two parameters, \( a_0 \) and \( a_1 \), by \( p(t) = a_0 + a_1 t \). Consider the points on the line \((t_i, p(t_i))\), and pick some vector norm \( \cdot \| \cdot \). Call the residuals \( r_i = y_i - p(t_i) \), and take \( r \) to be the vector of residuals. We would like to find the values \( a_0 \) and \( a_1 \) which give, for example,

\[ \min_{a_0, a_1} ||r||_2^2 = \min_{a, b} \sqrt{r_1^2 + r_2^2 + \ldots + r_n^2}. \]

The simplest norm to use is the 2-norm (which is why the alternate name for the method is least squares); then what we minimize is the following (picture).

Note that we could use any other norm we would like, and indeed in certain cases this makes sense; we could use \( \| \cdot \|_1 \) or \( \| \cdot \|_\infty \) as well. The name “least squares” is specific to using the 2-norm.
Matrix form. Note that the setup of the problem could be written as \( \min_{x \in \mathbb{R}} ||Ax - b||_2 \), where

\[
A = \begin{bmatrix}
1 & t_1 \\
1 & t_2 \\
\vdots & \vdots \\
1 & t_n
\end{bmatrix}, \quad x = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.
\]

Polynomials. Sometimes it may make sense to use a more complicated polynomial than a simple straight line; in that case, we will denote by

\[
p_m(t) = a_0 + a_1 t + \ldots + a_m t^m
\]
a degree \( m \) polynomial, and we will be interested in finding the values \( a_0, a_1, \ldots, a_m \) which minimize the residual \( r \) with \( r_i = y_i - p_m(t_i) \); often, we will use the Euclidean norm \( ||\cdot||_2 \).

Example. To set up a least squares problem where we would like to use a quartic polynomial \( p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \) to find the “best fit” to the set of points \((t_i, y_i)\) with \( i = 1, 2, \ldots, n \), we need to define the quantities

\[
A = \begin{bmatrix}
1 & t_1^2 & t_1^3 & t_1^4 \\
1 & t_2^2 & t_2^3 & t_2^4 \\
\vdots & \vdots & \vdots & \vdots \\
1 & t_n^2 & t_n^3 & t_n^4
\end{bmatrix}, \quad x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.
\]

The problem then becomes: find \( \min_{x \in \mathbb{R}^5} ||Ax - b||_2 \).

Finally, the set of functions we need to use to construct the fit or to interpolate does not have to be a polynomial. We could ask for the interpolating function to be a linear combination of exponentials, or sinusoidal functions, or really anything we want; once the set of functions to use is established, we will attach a coefficient to each of them and construct a linear combination which minimizes the 2-norm of the residual.