Mahowald invariants and the $\mathbb{R}$-motivic Adams spectral sequence

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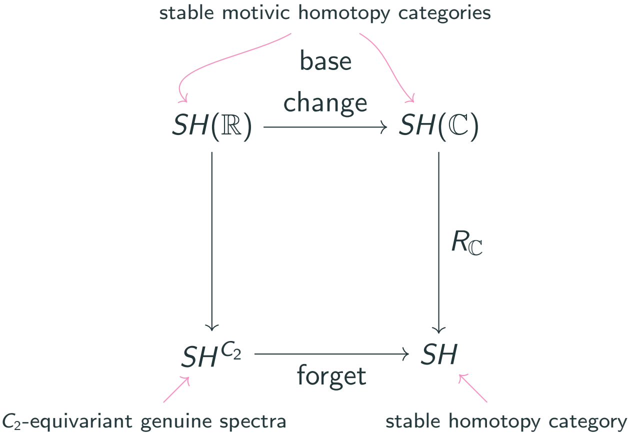
Joint work with Dan Isaksen

I. $\mathbb{R}$-motivic vs. $\mathbb{C}$-motivic homotopy groups of spheres

The main players



Motivic and C_2-equivariant spheres

$SH(k)$**: stable motivic category** $/k$
(Idea: simplicial ~~sets~~ k-schemes)

SH^{C_2}**: Genuine** C_2**-equivariant spectra**

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- 2 kinds of spheres:
  - Simplicial sphere
  - "Geometric sphere" $\mathbb{A}^1 - \{0\}$
- $S^{a,b} = (\text{simplicial sphere})^{\wedge a-b} \wedge (\text{geometric sphere})^{\wedge b}$
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- Bigraded homotopy π^k_{**}
- Realization functors for $k = \mathbb{C}, \mathbb{R}$:
  - $R_{\mathbb{C}} : SH(\mathbb{C}) \to SH$ (take $\mathbb{C}$-points)
  - $R_{\mathbb{R}} : SH(\mathbb{R}) \overset{\mathbb{C}\text{-points}}{\to} SH^{C_2} \overset{\Phi}{\to} SH$

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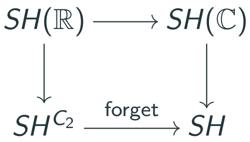
Motivic and C_2-equivariant spheres

$$\begin{array}{ccc}
SH(\mathbb{R}) & \xrightarrow{\text{base change}} & SH(\mathbb{C}) \\
\downarrow & & \downarrow R_{\mathbb{C}} \\
SH^{C_2} & \xrightarrow{\text{forget}} & SH \\
\downarrow \Phi & & \\
SH & &
\end{array}
\qquad R_{\mathbb{R}} : SH(\mathbb{R}) \to SH$$

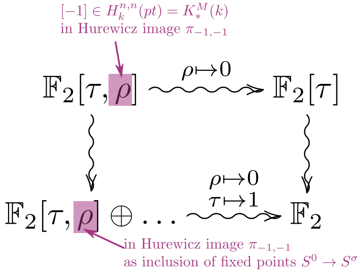
Realization of geometric sphere:

$$\begin{array}{ccc}
S^{1,1} & \longmapsto & S^{1,1} \\
\downarrow & & \downarrow \\
S^{\sigma} & \longmapsto & S^{1} \\
\downarrow & & \\
S^{0} & &
\end{array}$$

τ and ρ



Cohomology of a point:



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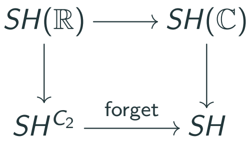

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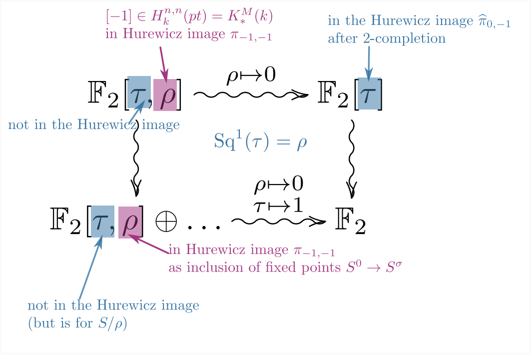

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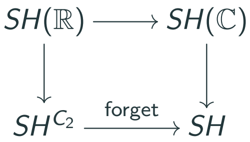

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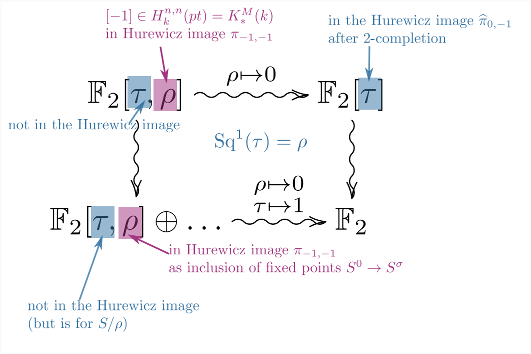

Slogan

- *$SH(\mathbb{C})$ is a deformation of classical homotopy theory*
- *$SH(\mathbb{R})$ captures some of the information in SH^{C_2}, but is easier to work with*

τ and ρ

$$\begin{array}{ccc} SH(\mathbb{R}) & \longrightarrow & SH(\mathbb{C}) \\ \downarrow & & \downarrow \\ SH^{C_2} & \xrightarrow{\text{forget}} & SH \end{array}$$

Cohomology of a point

$$\begin{array}{ccc} \mathbb{F}_2[\tau, \rho] & \overset{\rho \mapsto 0}{\rightsquigarrow} & \mathbb{F}_2[\tau] \\ \wr\downarrow & & \wr\downarrow \tau \mapsto 1 \\ \mathbb{F}_2[\tau, \rho] \oplus \dots & \overset{\substack{\rho \mapsto 0 \\ \tau \mapsto 1}}{\rightsquigarrow} & \mathbb{F}_2 \end{array}$$

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Adams spectral sequences:

$$E_1 = \text{Cobar complex on } (\mathbb{F}_2, A_*) \implies \pi_*(S)^{\wedge}_2$$

(in SH)

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$$\begin{array}{ccc} E_r(SH(\mathbb{R})) & \longrightarrow & E_r(SH(\mathbb{C})) \\ \downarrow & & \downarrow \\ E_r(SH^{C_2}) & \longrightarrow & E_r(SH) \end{array}$$

$\mathbb{R}$-motivic Adams spectral sequence: Main steps

- Compute the E_2 page $\mathrm{Ext}^{**}_{A^{\mathbb{R}}_*}(\mathbb{F}_2[\tau, \rho], \mathbb{F}_2[\tau, \rho])$

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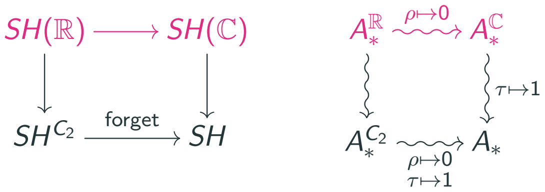

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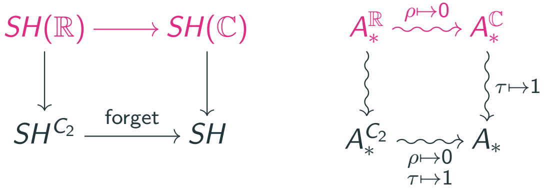

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  - Use ρ-Bockstein spectral sequence with $E_1 = \mathrm{Ext}_{A_*^{\mathbb{C}}}$

- Adams differentials
  - Comparison with $SH(\mathbb{C})$

Slogan: $SH(\mathbb{R})/\rho = SH(\mathbb{C})$

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- $\mathrm{Ext}_{A^{\mathbb{R}}}/\rho \cong \mathrm{Ext}_{A^{\mathbb{C}}}$
- There are maps of Adams spectral sequences

$$E_r(S^{\mathbb{R}}) \to E_r(S^{\mathbb{R}}/\rho) \overset{\cong}{\to} E_r(S^{\mathbb{C}})$$

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- $\pi^{\mathbb{R}}_{**}(S)/\rho \cong \pi^{\mathbb{C}}_{**}(S)$

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- $\pi_{**}^{\mathbb{R}}(S)/\rho \cong \pi_{**}^{\mathbb{C}}(S)$
- (Behrens-Isaksen-Shah-Xu) *After adding appropriate adjectives,* there is an equivalence of categories

$$(S^{\mathbb{R}}/\rho)\text{-modules} \simeq SH(\mathbb{C}).$$

Long exact sequence for $S \xrightarrow{\rho} S \to S/\rho$

$$\vdots$$
$$\downarrow$$
$$\pi^{\mathbb{R}}_{s+1,w+1}(S^{\mathbb{R}})$$
$$\downarrow \rho$$
$$\pi^{\mathbb{R}}_{s,w}(S^{\mathbb{R}})$$
$$\downarrow$$
$$\underbrace{\pi^{\mathbb{R}}_{s,w}(S^{\mathbb{R}}/\rho)}_{\cong \pi^{\mathbb{C}}_{s,w}(S)}$$
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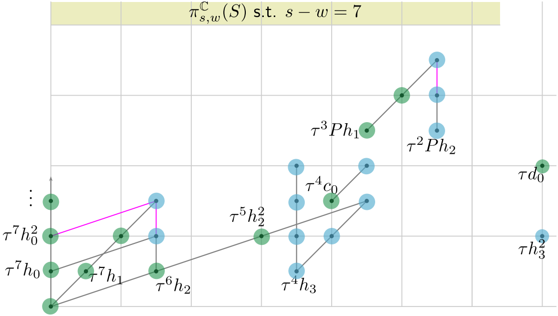

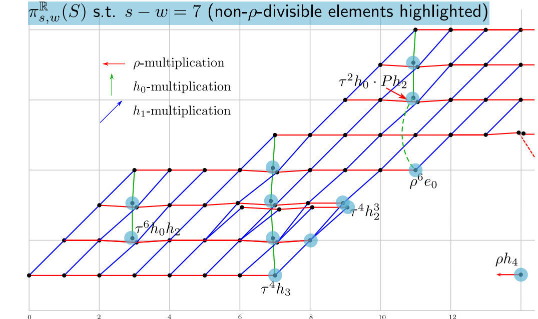

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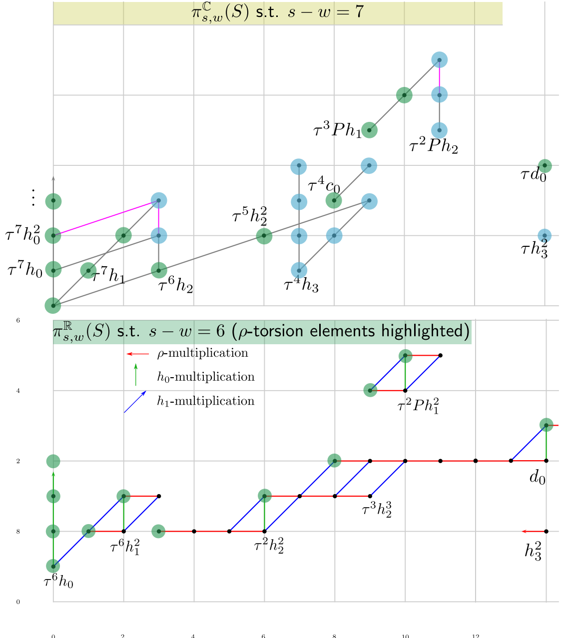
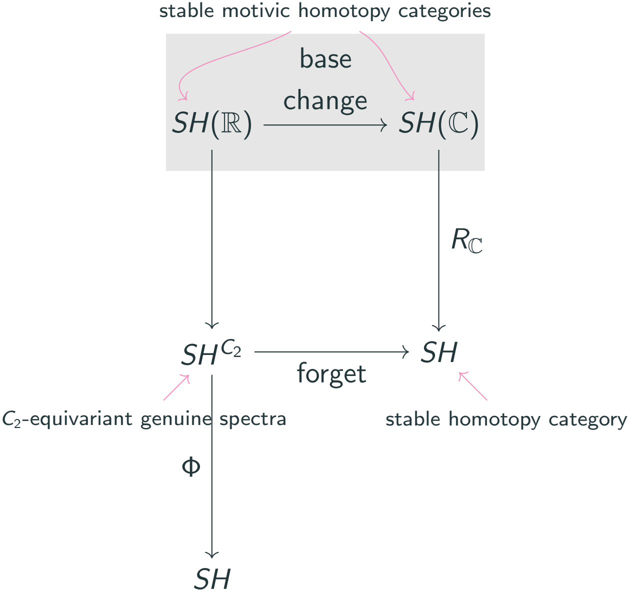


Uses of the relationship between $\pi^{\mathbb{C}}_{**}(S)$ and $\pi^{\mathbb{R}}_{**}(S)$

Sample arguments:

- **Map of spectral sequences $E_r(SH(\mathbb{R})) \to E_r(SH(\mathbb{C}))$ helps prove some $\mathbb{R}$-Adams differentials.**
  $d_3(h_0 h_4) = h_0 d_0$ over $\mathbb{C}$
  $d_3(h_0 h_4) = h_0 d_0$ or $h_0 d_0 + \rho h_1 d_0$ over $\mathbb{R}$
- **Showing hidden ρ-multiplications in homotopy:** x is non-ρ-divisible in E_∞, but it can't correspond to anything in $\pi^{\mathbb{C}}_{**}(S)$... so it has to be ρ-divisible in homotopy.
- **More Adams differentials:** $x \in \pi^{\mathbb{R}}_{**}(S)$ (non-ρ-divisible) has to be involved in an Adams differential, because it can't correspond to anything in $\pi^{\mathbb{C}}_{**}(S)$... and there's no chance of a hidden ρ-extension.

$SH(\mathbb{R})$ vs. SH



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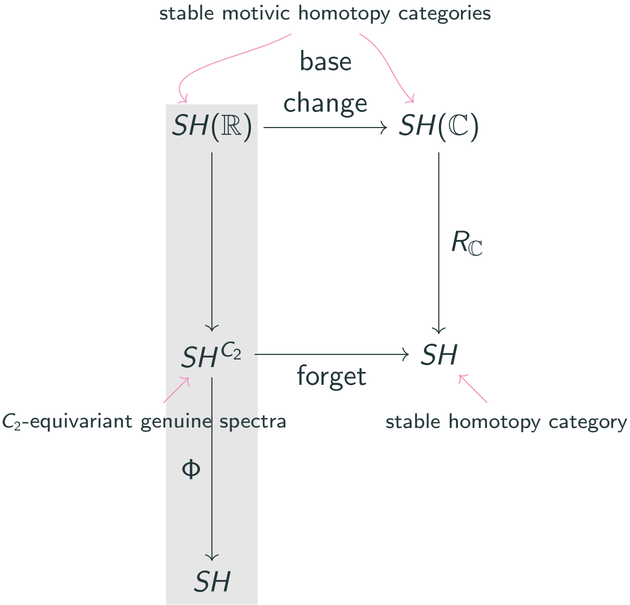

Some elements in $\pi_{**}^{\mathbb{R}}(S)$

$$R_{\mathbb{R}} : \pi^{\mathbb{R}}_{s,w}(S) \overset{\mathbb{C}\text{-points}}{\to} \pi^{C_2}_{s,w}(S) \overset{\Phi}{\to} \pi_{s-w}(S)$$

Theorems (Dugger-Isaksen)

- *There is an isomorphism* $\rho^{-1}\pi^{\mathbb{R}}_{**}(S) \cong \pi_*(S)[\rho^{\pm 1}]$.
  On Adams E_2 *pages, this is given by* Sq^0.
- $\widehat{\pi}^{\mathbb{R}}_{s,w}(S) \cong \widehat{\pi}^{C_2}_{s,w}(S)$ *if* $s \geq 3w - 5$.

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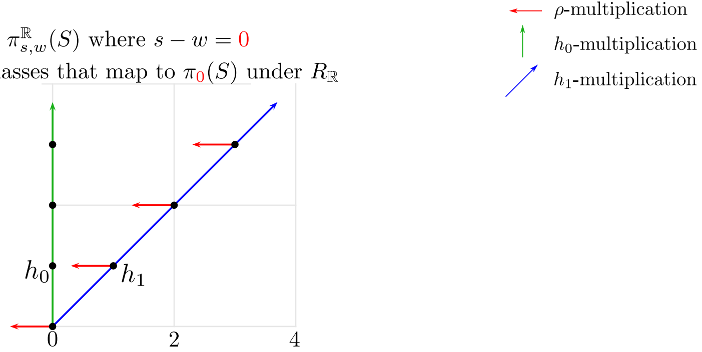
$\pi^{\mathbb{R}}_{s,w}(S)$ where $s - w = 0$

Classes that map to $\pi_0(S)$ under $R_{\mathbb{R}}$



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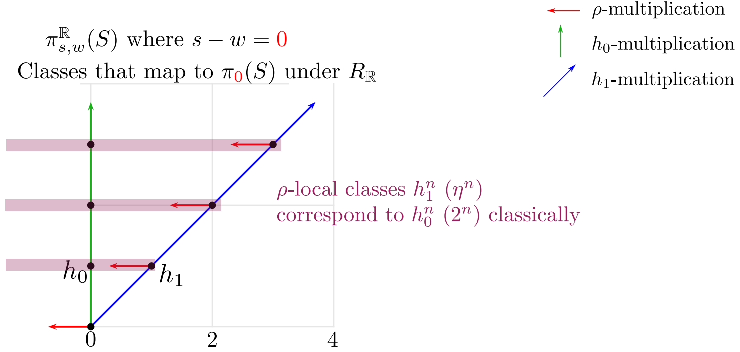

ρ-local classes h_1^n (η^n) correspond to h_0^n (2^n) classically

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← ρ-multiplication

↑ h_0-multiplication

↗ h_1-multiplication

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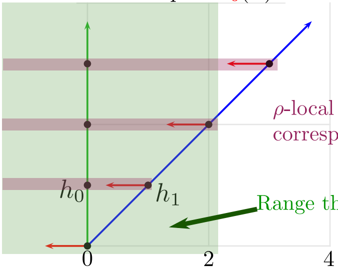

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correspond to h_0^n (2^n) classically

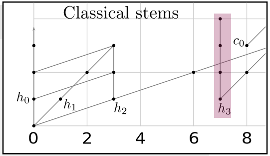
Range that agrees with π^{C_2}

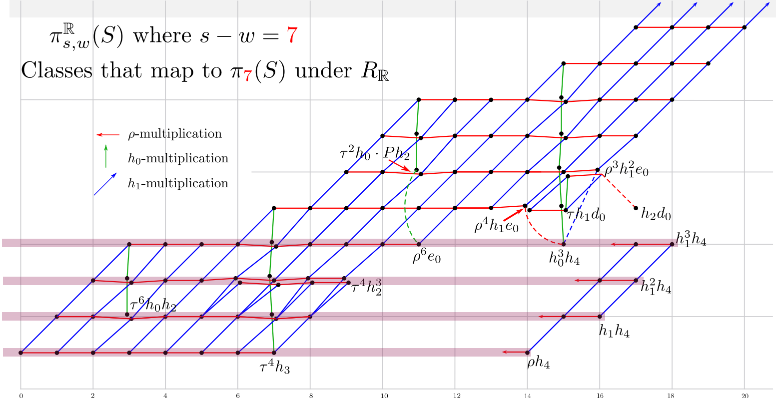
Some elements in $\pi^{\mathbb{R}}_{**}(S)$

$$R_{\mathbb{R}} : \pi^{\mathbb{R}}_{s,w}(S) \overset{\mathbb{C}\text{-points}}{\to} \pi^{C_2}_{s,w}(S) \overset{\Phi}{\to} \pi_{s-w}(S)$$

Theorems (Dugger-Isaksen)

- *There is an isomorphism $\rho^{-1}\pi^{\mathbb{R}}_{**}(S) \cong \pi_*(S)[\rho^{\pm 1}]$. On Adams E_2 pages, this is given by Sq^0.*
- $\widehat{\pi}^{\mathbb{R}}_{s,w}(S) \cong \widehat{\pi}^{C_2}_{s,w}(S)$ *if $s \geq 3w - 5$.*





Some elements in $\pi^{\mathbb{R}}_{**}(S)$

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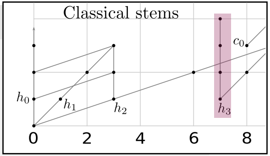
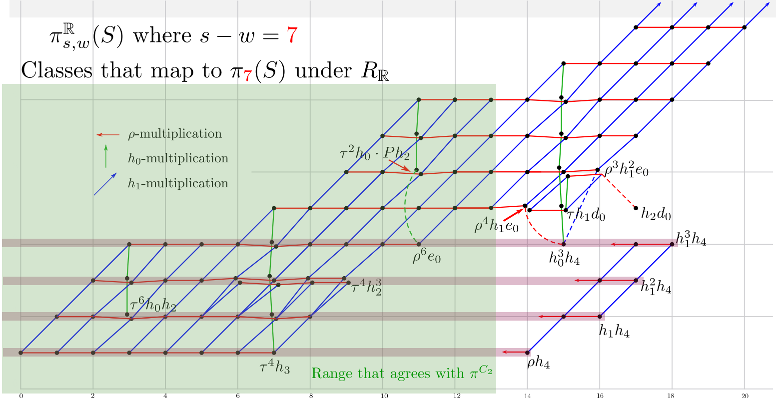


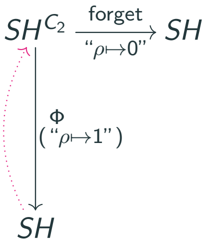


II. Mahowald Invariant

Mahowald invariant

$\pi_*(S) \to \pi_*(S)$ (with indeterminacy)

Disclaimer: this is a small lie (it has too much indeterminacy)



Mahowald invariant

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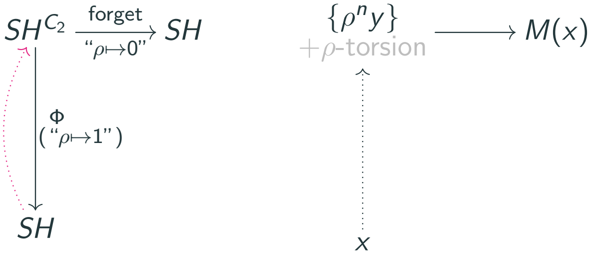
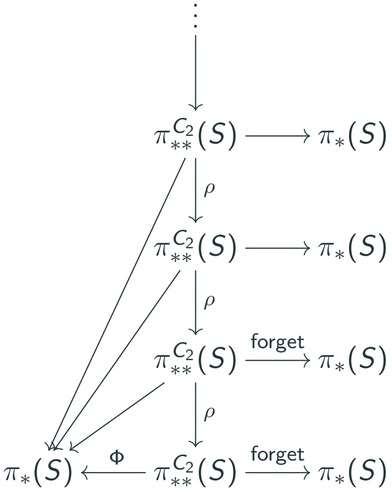


Theorem (Bredon, Araki-Iriye)

There is an isomorphism $\rho^{-1}\pi_{**}^{C_2}(S) \cong \pi_*(S)[\rho^{\pm 1}]$.

Mahowald invariant

$$
\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
\pi^{C_2}_{**}(S) & \longrightarrow & \pi_*(S) \\
\downarrow{\scriptstyle \rho} & & \\
\pi^{C_2}_{**}(S) & \longrightarrow & \pi_*(S) \\
\downarrow{\scriptstyle \rho} & & \\
\pi^{C_2}_{**}(S) & \xrightarrow{\text{forget}} & \pi_*(S) \\
\downarrow{\scriptstyle \rho} & & \\
\pi^{C_2}_{**}(S) & \xrightarrow{\text{forget}} & \pi_*(S)
\end{array}
$$

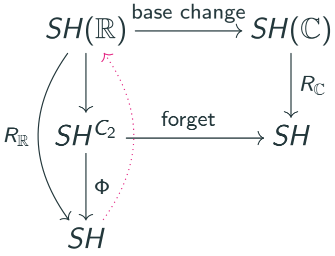

$\pi_{**}^{C_2}(S)$
$\pi_*(S)$
ρ
$\pi_{**}^{C_2}(S)$
$\pi_*(S)$
ρ
$\pi_{**}^{C_2}(S)$
forget
$\pi_*(S)$
ρ
$\pi_*(S)$
Φ
$\pi_{**}^{C_2}(S)$
forget
$\pi_*(S)$

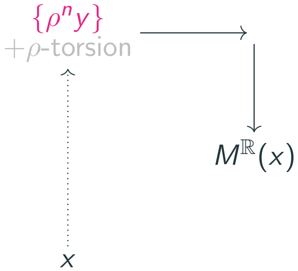
$M(x)$

x $\rho^2 y$

A variation involving $SH(\mathbb{R})$

Disclaimer: this is also a small lie (it has too much indeterminacy)





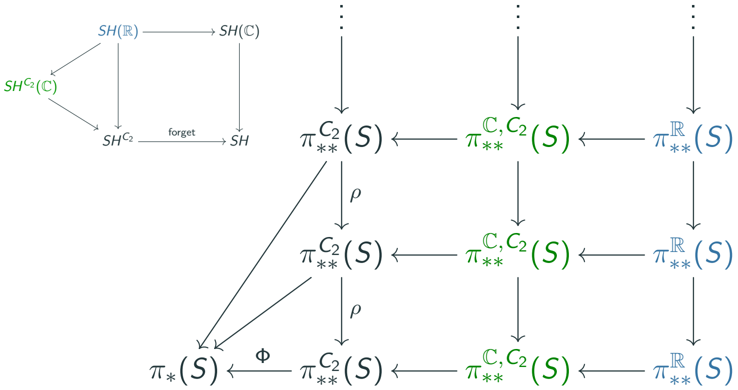
Theorem (Dugger-Isaksen)

There is an isomorphism $\rho^{-1}\pi^{\mathbb{R}}_{**}(S) \cong \pi_*(S)[\rho^{\pm 1}]$.

On Adams E_2 *pages, this is given by* Sq^0.

A variation involving $SH(\mathbb{R})$

- Our work: lift to $SH(\mathbb{R})$ tower
- Quigley: a different variant of the Mahowald invariant (defined on $\pi^{\mathbb{C}}_{**}(S)$ instead of $\pi_*(S)$) from lifting to the $SH^{C_2}(\mathbb{C})$ tower



A variation involving $SH(\mathbb{R})$

Question

When does this recover the classical Mahowald invariant?

A variation involving $SH(\mathbb{R})$

Question

When does this recover the classical Mahowald invariant?

Answer: At least in some interesting cases, but not always.

Mahowald invariant of 2 is η

Slogan

The "default" Mahowald invariant is given by Sq^0.

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ & & \pi^{C_2}_{**}(S) & \longrightarrow & \pi_*(S) \\ & & \downarrow \rho & & \\ & & \pi^{C_2}_{**}(S) & \longrightarrow & \pi_*(S) \\ & & \downarrow \rho & & \\ \pi_*(S) & \longleftarrow & \pi^{C_2}_{**}(S) & \longrightarrow & \pi_*(S) \end{array}$$

$2 \dashrightarrow \eta$

$$\{\rho^n h_0\} \xrightarrow{\mathrm{Sq}^0} \{\rho^n h_1\}$$

$$E_2(\text{classical})[\rho^{\pm}] \cong \rho^{-1} E_2(C_2)$$

$$\Downarrow \text{Adams} \qquad\qquad \Downarrow \text{Adams}$$

$$\pi_*(S)[\rho^{\pm}] \cong \rho^{-1}\pi^{C_2}_{**}(S)$$

$$\{\rho^n 2\} \dashrightarrow \{\rho^n \eta\}$$

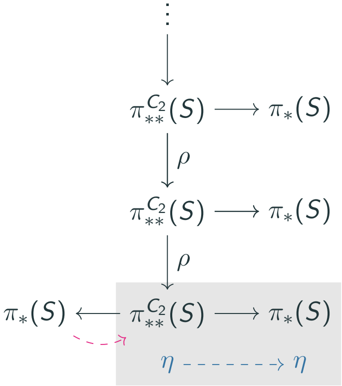
$$\pi^{C_2}_{**}(S) \longrightarrow \rho^{-1}\pi^{C_2}_{**}(S)$$

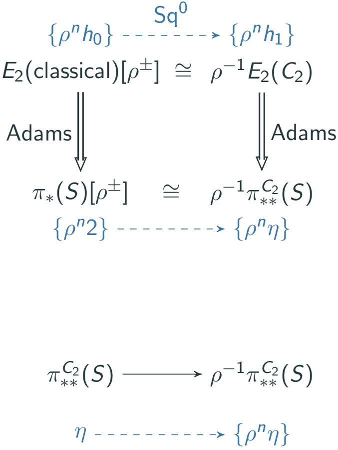
$$\eta \dashrightarrow \{\rho^n \eta\}$$

Mahowald invariant of 2 is η

Slogan

The "default" Mahowald invariant is given by Sq^0.





Mahowald invariant of σ is σ^2 (via $SH(\mathbb{R})$) $\quad (\sigma \longleftrightarrow h_3)$

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & \downarrow \rho & & \\
 & & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & \downarrow \rho & & \\
\pi_*(S) & \longleftarrow & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
\sigma & \dashrightarrow & ?? & &
\end{array}
$$

$$
\begin{array}{ccc}
\{\rho^n h_3\} & \overset{\mathrm{Sq}^0}{\dashrightarrow} & \{\rho^n h_4\} \\
E_2(\text{classical})[\rho^{\pm}] & \cong & \rho^{-1} E_2(\mathbb{R}) \\
\text{Adams} \Downarrow & & \Downarrow \text{Adams} \\
\pi_*(S)[\rho^{\pm}] & \cong & \rho^{-1}\pi^{\mathbb{R}}_{**}(S) \\
\{\rho^n \sigma\} & \dashrightarrow & \{\rho^n [h_4]\}
\end{array}
$$

$$
\begin{array}{ccc}
\pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \rho^{-1}\pi^{\mathbb{R}}_{**}(S) \\
?? & \dashrightarrow & \{\rho^n [h_4]\}
\end{array}
$$

Mahowald invariant of σ is σ^2 (via $SH(\mathbb{R})$) $\quad (\sigma \longleftrightarrow h_3)$

$$\vdots$$

$$\downarrow$$

$$\pi^{\mathbb{R}}_{**}(S) \longrightarrow \pi_*(S)$$

$$\downarrow \rho$$

$$\pi^{\mathbb{R}}_{**}(S) \longrightarrow \pi_*(S)$$

$$\downarrow \rho$$

$$\pi_*(S) \longleftarrow \pi^{\mathbb{R}}_{**}(S) \longrightarrow \pi_*(S)$$

$$\sigma \dashrightarrow ??$$

$$\{\rho^n h_3\} \overset{\mathrm{Sq}^0}{\dashrightarrow} \{\rho^n h_4\}$$

$$E_2(\text{classical})[\rho^{\pm}] \cong \rho^{-1}E_2(\mathbb{R})$$

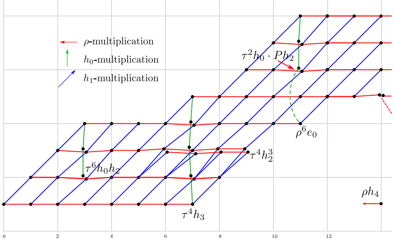
$$\Downarrow \text{Adams} \qquad \Downarrow \text{Adams}$$

$$\pi_*(S)[\rho^{\pm}] \cong \rho^{-1}\pi^{\mathbb{R}}_{**}(S)$$

$$\{\rho^n \sigma\} \dashrightarrow \{\rho^n [h_4]\}$$

$$\pi^{\mathbb{R}}_{**}(S) \longrightarrow \rho^{-1}\pi^{\mathbb{R}}_{**}(S)$$

$$?? \dashrightarrow \{\rho^n [h_4]\}$$



Mahowald invariant of σ is σ^2 (via $SH(\mathbb{R})$) $\quad (\sigma \longleftrightarrow h_3)$

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & \downarrow \rho & & \\
 & & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & \downarrow \rho & & \\
\pi_*(S) & \longleftarrow & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S)
\end{array}
$$

$\sigma \dashrightarrow [\rho h_4]$

$$\{\rho^n h_3\} \overset{\mathrm{Sq}^0}{\dashrightarrow} \{\rho^n h_4\}$$

$$E_2(\text{classical})[\rho^{\pm}] \cong \rho^{-1} E_2(\mathbb{R})$$

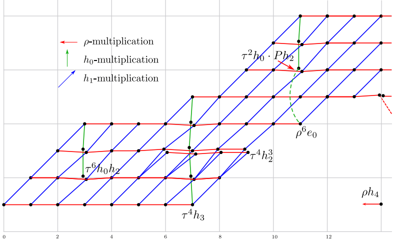
$$\Downarrow \text{Adams} \qquad \Downarrow \text{Adams}$$

$$\pi_*(S)[\rho^{\pm}] \cong \rho^{-1}\pi^{\mathbb{R}}_{**}(S)$$

$$\{\rho^n \sigma\} \dashrightarrow \{\rho^n [h_4]\}$$

$$\pi^{\mathbb{R}}_{**}(S) \longrightarrow \rho^{-1}\pi^{\mathbb{R}}_{**}(S)$$

$$[\rho h_4] \dashrightarrow \{\rho^n [h_4]\}$$



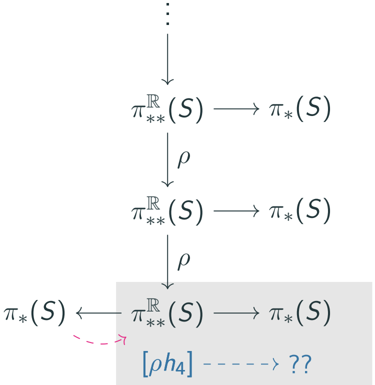
Mahowald invariant of σ is σ^2 (via $SH(\mathbb{R})$) $(\sigma \longleftrightarrow h_3)$

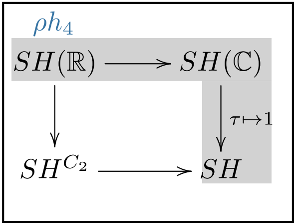
$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & \downarrow \rho & & \\
 & & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & \downarrow \rho & & \\
\pi_*(S) & \longleftarrow & \pi^{\mathbb{R}}_{**}(S) & \longrightarrow & \pi_*(S) \\
 & & [\rho h_4] & \dashrightarrow & ??
\end{array}
$$

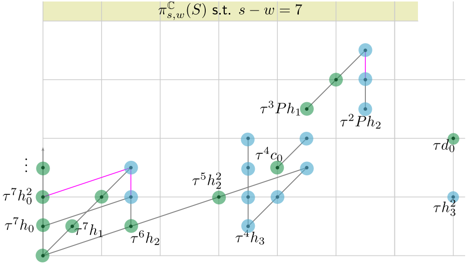
ρh_4

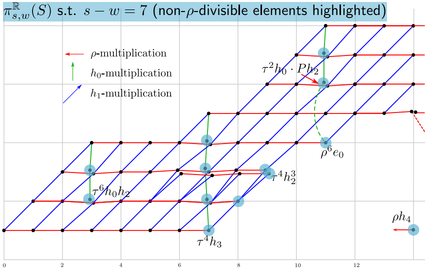
$$
\begin{array}{ccc}
SH(\mathbb{R}) & \longrightarrow & SH(\mathbb{C}) \\
\downarrow & & \downarrow {\scriptstyle \tau \mapsto 1} \\
SH^{C_2} & \longrightarrow & SH
\end{array}
$$

Mahowald invariant of σ is σ^2 (via $SH(\mathbb{R})$) $(\sigma \longleftrightarrow h_3)$









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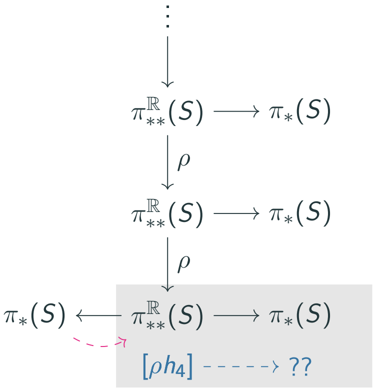
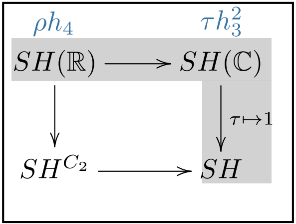
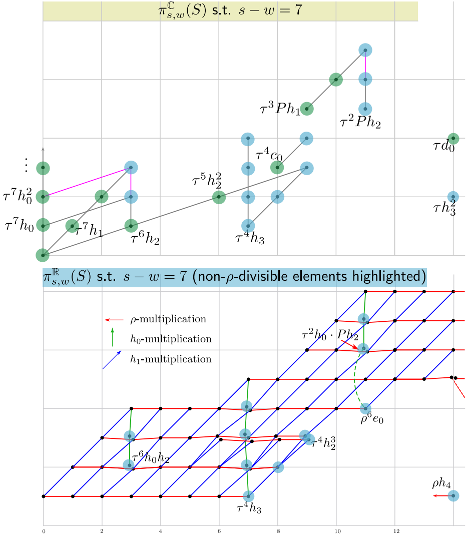
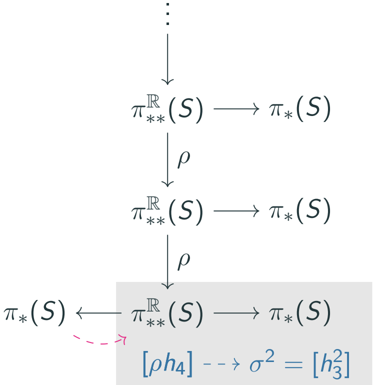
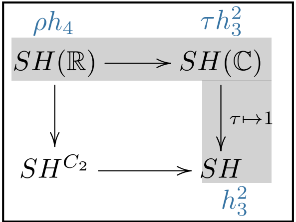


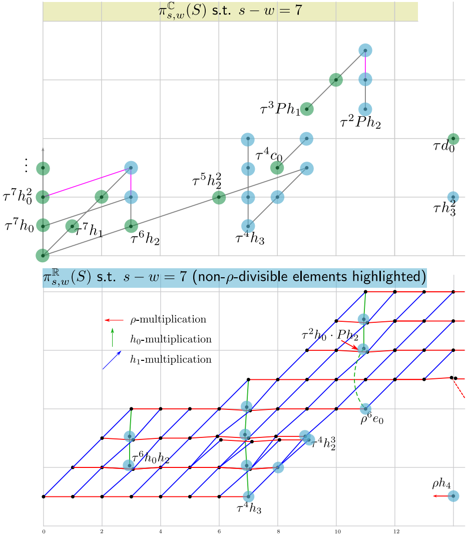




Mahowald invariant of σ is σ^2 (via $SH(\mathbb{R})$) $\quad (\sigma \longleftrightarrow h_3)$







The end.