Simplicial Sets and One Notion of ∞-Categories

Eva Belmont
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There are lots of approaches to defining infinity categories. In this talk, I will give the necessary background for explaining one such construction, namely *quasicategories*.

1. From simplicial complexes to simplicial sets

(Ordered) Simplicial complexes. Simplicial complexes are basically things you can build by gluing together *standard ordered n-simplices* along their faces.

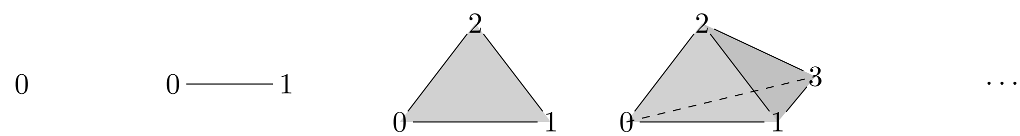
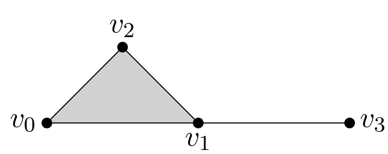


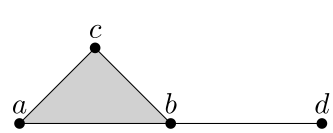
Figure 1. Standard n-simplices

For example, this



is a simplicial complex. (Note that for the purposes of this talk, simplicial complexes will be ordered – that is, there is an ordering on the vertices.) You can specify a simplicial complex by labelling the vertices $\{v_1, v_2, \cdots, v_n\}$ and specifying several sets:

$$X_0 = \{ \text{ vertices } \}$$
$$X_1 = \{ \text{ edges } \}$$
$$X_2 = \{ \text{ faces } \} \quad \text{etc.}$$

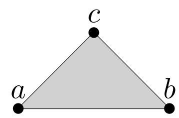
So if we label the simplicial complex above as 

then $X_0 = \{a, b, c, d\}$, $X_1 = \{(a, b), (b, c), (c, a), (b, d)\}$, and X_2 is the set containing just the one 2-dimensional face (a, b, c). Note that all higher X_i are empty.

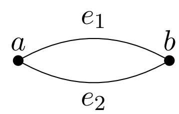
This kind of description has a lot of redundancy. For every n, and $0 \le i \le n$, you can form maps of sets $d_i : X_{n+1} \to X_n$, called *face maps*, that take an n-simplex to the i^{th} face. It is not hard to check that these satisfy the identity

$$d_i d_j = d_{j-1} d_i \text{ for } i < j. \tag{1.1}$$

Delta complexes. You can think of delta complexes as the same as simplicial complexes, only you allow multigraphs. More formally, a delta complex is a collection of sets $X_0, X_1, X_2, \cdots$ as before, and face maps $d_i : X_{n+1} \to X_n$ satisfying (1.1). For example, simplicial complexes are delta complexes, and so is anything made by taking a simplicial complex and gluing some of its simplices to each other. For example, we can take



and glue a to b to get a cone. For another example, consider

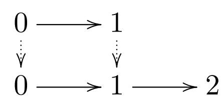


, where the only nonempty sets are $X_0 = \{a, b\}$ and $X_1 = \{e_1, e_2\}$.

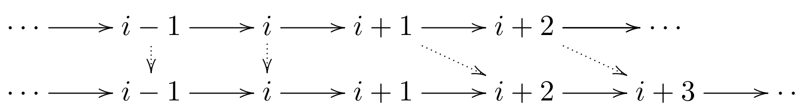
Another way to describe a delta complex is as a functor $X : \widehat{\Delta}^{op} \to \mathbf{Set}$, where $\widehat{\Delta}$ is a category where the objects are

$$\underline{n} = 0 \to 1 \to \cdots \to n$$

for every n. On objects, this sends $\underline{n} \mapsto X_n$, and morphisms are strictly order-preserving maps. So a morphism $\underline{1} \to \underline{2}$ might look like

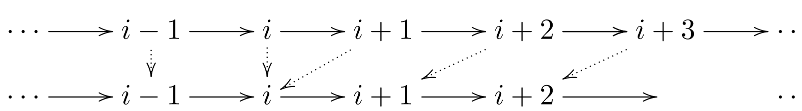


and there can be no morphisms $\underline{2} \to \underline{1}$ (because then two things in $\underline{2}$ would have to be sent to one thing in $\underline{1}$). We can describe our face maps d_i in terms of this language: there is a morphism $d^i : \underline{n} \to \underline{n+1}$



which is sent to the face map $d_i : X_n \to X_{n+1}$ under $X : \widehat{\Delta}^{op} \to \mathbf{Set}$. In this reformulation, you can again check that (1.1) holds.

Simplicial sets. Simplicial sets are like delta complexes, except the morphisms are only *order-preserving*, not necessarily *strictly* order-preserving. So we have a new category Δ, where the objects are $\underline{n}$ as before, and the morphisms are order-preserving maps. Now we have *degeneracies* $s^i : \underline{n+1} \to \underline{n}$ as follows:



The functor $X : \Delta^{op} \to \mathbf{Set}$ sends s^i to a map $s_i : X_n \to X_{n+1}$ satisfying

$$s_i s_j = s_{j+1} \qquad \text{for } i \leq j$$

$$d_i s_j = \begin{cases} 1 & \text{if } i = j \text{ or } i = j+1 \\ s_{j-1} d_i & \text{if } i < j \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

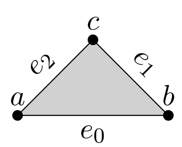
Let's take a look at what happens to the most basic geometric notions under the categorical lens. For every n the analogue of the standard n-simplex (as in Figure 1) is a simplicial set

$\Delta^n = \mathrm{Hom}_\Delta(-, \underline{n})$ (that is, $(\Delta^n)_k = \mathrm{Hom}(\underline{k}, \underline{n})$). So $(\Delta^n)_0$ is the set $\mathrm{Hom}(\underline{0}, \underline{n})$, which corresponds to picking an element i in $0 \to \cdots \to n$. Similarly, $(\Delta^n)_1 = \mathrm{Hom}(\underline{1}, \underline{n})$ corresponds to picking a pair (i, j) for $i \leq j$ – that is, picking a path $v_i \to v_j$.

By the Yoneda lemma, there is a correspondence $\{\Delta^n \to X\} \leftrightarrow X_n$. If X is a simplicial set that came from a simplicial complex, then we may think of X_n as the set of (combinatorial) n-simplices in the simplicial complex (including degeneracies). This also motivates the following definition.

Definition 1.1. The n-simplices of X are simplicial maps $\Delta^n \to X$.

Example 1.2. The simplicial complex



corresponds to a simplicial set $X : \Delta^{op} \to \mathbf{Set}$ where $X_0 = \{a, b, c\}$, $X_1 = \{e_0, e_1, e_2, s_0(a), s_0(b), s_0(c)\}$, and X_{100} contains $s_0(a), s_0(b), s_0(c)$ (maps $[0, \cdots, 100] \mapsto a$ etc.), $s_i(e_1), s_i(e_2), s_i(e_3)$ (maps $[0, \cdots, i] \mapsto a$ and $[i+1, \cdots, 100 \mapsto b]$ etc.).

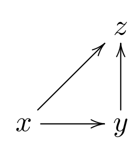
Example 1.3. Let W be a topological space. Then we will define a simplicial set $\mathrm{Sing}\, W$ as:

$$(\mathrm{Sing}\, W)_n = \{|\Delta^n| \to W\}$$

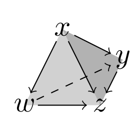
where $|\Delta^n|$ is the *geometric* n-simplex – the figure in $\mathbb{R}^n$ with vertices $v_i = (\underbrace{0, \cdots, 0}_{i}, 1, 0, \cdots, 0)$, for $0 \leq i \leq n$. The face maps and degeneracies are what you'd expect geometrically.

Example 1.4 (Nerve)**.** Let $\mathcal{C}$ be an arbitrary category. We will define a simplicial set $B\mathcal{C}$ called the *nerve*.

$$(B\mathcal{C})_0 = \{ \text{ objects of } \mathcal{C} \}$$
$$(B\mathcal{C})_1 = \{ \text{ morphisms of } \mathcal{C} \}$$
$$(B\mathcal{C})_2 = \left\{ \text{commutative diagrams} \right\}$$



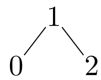
$$(B\mathcal{C})_3 = \left\{ \text{commutative tetrahedrons} \right\}$$



$$(B\mathcal{C})_4 = \text{ etc.}$$

2. Kan condition

Definition 2.1 (Horn)**.** For all $k \leq n$, form the simplicial set Λ^n_k, called the "(n, k) *horn*" by taking Δ^n and removing the n-simplex and its k^{th} face.

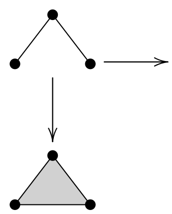
For example, Λ^2_0 is  .

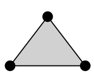
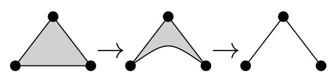
Definition 2.2. A simplicial set X has the *Kan condition* if, for every diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & & \\ \Delta^n & & \end{array} \tag{2.1}$$

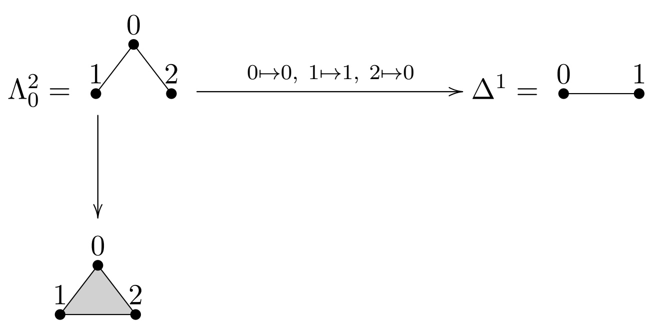
there is an extension $\Delta^n \to X$ making the diagram commute. (The Kan condition is informally summarized as "every horn has a filler".)

Example 2.3. $\operatorname{Sing} W$ satisfies the Kan condition. We can illustrate this in the case $n = 2$: the diagram (2.1) looks like



and the filler  $\to X$ is given by contracting  before mapping that to X.

Example 2.4 (Δ^1)**.** We will show that Δ^1 doesn't have the Kan condition by constructing a horn diagram (2.1) that doesn't have a filler. Consider



(along the top row we're mapping the edge $(0, 1)$ to $(0, 1)$ and the edge $(0, 2)$ to the degenerate edge $(1, 1)$).

Example 2.5. Let's check if the nerve $B\mathcal{C}$ satisfies the Kan condition. First suppose the top row of (2.1) looks like $\Lambda^2_1 \to B\mathcal{C}$; this corresponds to picking objects c_0, c_1, c_2 in $B\mathcal{C}$ with morphisms $c_0 \to c_1$ and $c_1 \to c_2$. A map $\Delta^2 \to B\mathcal{C}$ would be a collection of three objects c_0, c_1, c_2 with morphisms $c_0 \to c_1, c_1 \to c_2, c_0 \to c_2$ that form a commutative triangle. So given the data $c_0 \to c_1 \to c_2$ of a map $\Lambda^2_1 \to B\mathcal{C}$, we can form a commutative triangle by defining $c_0 \to c_2$ to be the composition of $c_0 \to c_1 \to c_2$. So, any embedding of the horn Λ^2_1 has a filler.

However, now let's try to get a filler for an embedding of the horn Λ^2_0. A map $\Lambda^2_0 \to B\mathcal{C}$ is equivalent to picking objects c_0, c_1, c_2 and morphisms $c_0 \to c_1$ and $c_0 \to c_2$. In order to get a "filler" $\Delta^2 \to B\mathcal{C}$ we'd need to get another morphism $c_1 \to c_2$ making $\begin{array}{ccc} & c_1 & \\ \nearrow & & \searrow \\ c_0 & \longrightarrow & c_2 \end{array}$ commute.

Such a morphism might not necessarily exist, and so $B\mathcal{C}$ does not satisfy the Kan condition.

Note that this *does* work if $\mathcal{C}$ is a groupoid – that is, if all morphisms are invertible.

To get a filler, we need to send the edge $(1, 2)$ to some (possibly degenerate) edge. But there is no way to do this in an order-preserving manner.

In fact, $\Lambda^n_k \to B\mathcal{C}$ works when $0 < k < n$. If $0 < k < n$ then Λ^n_k is called an *inner horn.*

Definition 2.6. A quasi-category is a simplicial set where every inner horn has a filler.

If $\mathcal{C}$ is a category then inner horn fillers for $B\mathcal{C}$ are *unique*: this is because compositions are unique.

Proposition 2.7. If X has unique fillers of inner horns, then $X = B\mathcal{C}$ for some category $\mathcal{C}$.

Proof. To construct $\mathcal{C}$, let X_0 be the set of objects of $\mathcal{C}$, and X_1 be the set of morphisms. We can recover the source and destination of each morphism by the face maps: if $e \in X_1$ then we can think of e as a morphism from $d_0(e)$ to $d_1(e)$.

Identity morphisms: for $v \in X_0$, the identity morphism is $s_0(v)$.

Compositions: suppose we have morphisms $a : x \to y$ and $b : y \to z$. Then the composition $b \circ a$ is the unique filler in the diagram

$$\begin{array}{ccc} \Lambda^2_1 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array}$$

where the top row embeds Λ^2_1 as $\begin{array}{ccc} & y & \\ \nearrow & & \searrow \\ x & & z. \end{array}$

Associativity: a nuisance; use a similar idea with Λ^3_2. □

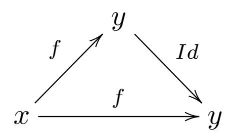
What if fillers are not unique? The idea is that, in a quasicategory, all possible fillers are "equivalent".

Definition 2.8. If $f, g : x \to y$ are 1-simplices, then we say that f is homotopic to g $(f \sim g)$ if there is a 2-simplex whose boundary is

$$\begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{Id} \\ x & \xrightarrow{\;g\;} & y \end{array}$$

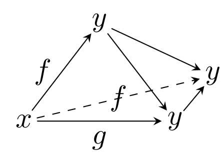
Proposition 2.9. If X is a quasicategory, then $\sim$ is an equivalence relation.

Proof. (Reflexivity) Let $f : x \to y$ and take the degenerate 2-simplex sending $0 \mapsto x, 1 \mapsto y, 2 \mapsto y$:



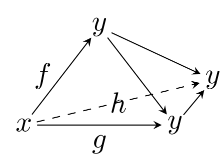
This shows that $f \sim f$.

(Symmetry) Suppose $f \sim g$; this implies the front face in the following diagram commutes (unlabelled arrows are the identity).



Then the right and back faces commute by definition of the identity arrow. By the inner-horn-filling condition, the entire tetrahedron commutes, and hence the bottom face commutes; now by definition, $g \sim f$.

(Transitivity) Suppose $f \sim g$ and $g \sim h$. This implies that the front and bottom faces commute, and the right face commutes by definition.

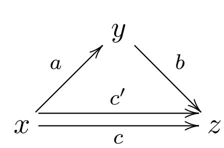


By the inner-horn-filling condition, $f \sim h$. ☐

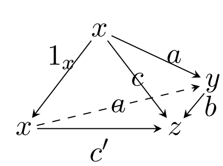
Proposition 2.10. In a quasicategory, two inner-horn-fillers are homotopic.

Proof. Given two fillers

(2.2)



it suffices to show that the front face of



commutes. But the right and bottom faces commute because of our assumption (2.2), and the back face commutes by definition. By the inner-horn-filling condition, the front face commutes. □

So given a generic quasicategory X, we might not have $X = B\mathcal{C}$ for some $\mathcal{C}$, but you can still associate a category with it: define $Ho(X)$ to be the category where the set of objects is X_0, and the morphisms are equivalence classes (up to homotopy) of elements in X_1.

References: Finster, Eric. Lecture notes on infinity categories, `http://sma.epfl.ch/~finster/infcats.html`, 2011.

Friedman, Greg. "An elementary illustrated introduction to simplicial sets", `http://arxiv.org/abs/0809.4221`, 2008.

Goerss, Paul and John Jardine. *Simplicial Homotopy Theory*, Birkhauser, 1999.

Tanaka, Hiro Lee. "An introduction to infinity-categories", `http://pages.uoregon.edu/njp/tanaka.pdf`, 2012.