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Stable Homotopy and the J-Homomorphism

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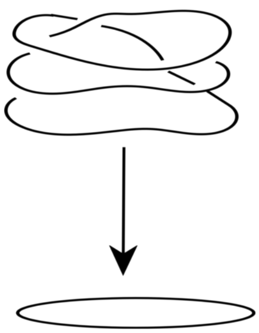
Chapter 1

Introduction to K-theory and the J-homomorphism

One of topology's earliest goals was to study the degree of maps $S^n \to S^n$: the number of times that the first copy of S^n winds around the second.



The dawn of algebraic topology brought the insight that this collection of maps could be given the structure of a group which is isomorphic to $\mathbb{Z}$: each $k \in \mathbb{Z}$ corresponds to a map $S^n \to S^n$ of degree k. This is not hard to see, once the idea has been made precise, so now you may want to ask the natural follow-up question: how do we classify maps $S^n \to S^m$ for any n, m? Half of this is again easy: if $n < m$, there are no such maps, other than (up to equivalence) the trivial map that collapses the entire domain to a point. Unfortunately, if $n > m$, this question is so hard that topologists are still stumped.

Our knowledge is frustratingly spotty. The groups $\pi_{n+1}S^n$ and $\pi_{n+2}S^n$ have been known since the 1940's (see [23]), but $\pi_n S^2$ remains unclassified. We know that $\pi_{n+k}(S^n)$ are finite, except for $\pi_{4m-1}S^{2n}$, which are finitely generated. The groups up through $\pi_{19+n}S^n$ are known in entirety; more have been calculated, and for still others, only certain prime components are known.

However, there is a range of groups that are more tractable. If $n \geq i+1$, then $\pi_{i+n}S^n$ is independent of n; these are called the *stable groups*, and are denoted π_i^s. We still do not know the full story of the stable groups, but we know a piece of it: this thesis tells part of that story. The exposition here is based primarily on Adams' papers [2]-[5], filtered through the insight of my advisor Prof. Michael Hopkins, who taught me this material.

Using K-theory over $\mathbb{R}$, it is possible to compute explicitly a cyclic group that is a direct summand of every π_i^s. Here we will work over $\mathbb{C}$ instead, which dodges some technicalities; the price is that we land a factor of 2 away in one computation from proving that there is a splitting.

§1.1 The Hopf fibration

As a starting point for studying $\pi_n(S^m)$, let us take a nontrivial example in low dimension:

the *Hopf fibration* $S^3 \to S^2$. View S^3 as a subset of $\mathbb{R}^4 \cong \mathbb{C}^2$, and use the fact that $S^2 \cong \mathbb{C}P^1$. The natural quotient map $\mathbb{C}^2 \to \mathbb{C}P^1$ induces a natural map $S^3 \to S^2$. The preimage of $[x, y] \in \mathbb{C}P^1$ is the set

$$\begin{aligned} \{(\lambda x, \lambda y) \,:\, |(\lambda x, \lambda y)| = 1\} &\cong \left\{\lambda \,:\, |\lambda|^2 = (|x|^2 + |y|^2)^{-1}\right\} \\ &\cong \mathbb{C}^* \cong S^1 \;. \end{aligned}$$

We say that S^1 is the fiber, and $S^1 \to S^3 \to S^2$ is a fibration; we will return to fiber maps later.

Alternatively, recall that we can decompose $\mathbb{C}P^2$ as the open 4-cell $\{[1, a, b] \,:\, a, b \in \mathbb{C}\}$, the open 2-cell $\{[0, 1, c] \,:\, c \in \mathbb{C}\}$, and the "point at infinity" $[0, 0, 1]$. The boundary of the 4-cell is the limit of circles $\{[1, Ra, Rb] \,:\, |a|^2 + |b|^2 = 1\} \cong \{(Ra, Rb)\} \in R \cdot S^3$ as $R \to \infty$. So the attaching map $A : \partial(\text{4-cell}) = \partial\{\ (a, b) \,:\, a, b \in \mathbb{C}\ \} \to \{[0, a, b]\}$ is homotopic to the limit of the maps $A_R : (Ra, Rb) \to [0, Ra, Rb]$. Each A_R is homotopic to the Hopf fibration defined above, and so $A = \lim_{R\to\infty} A_R$ is also just the Hopf fibration.

Other than being an easy example of a nontrivial element of $\pi_m(S^n)$ for $m > n$, the Hopf fibration is also worthy of consideration because it is the $n = 2$ case of the *Hopf construction*, a method for obtaining maps $H_f : S^{2n-1} \to S^n$, given the initial data of a map $f : S^{n-1} \times S^{n-1} \to S^{n-1}$. The reason why anyone would care about maps $S^{n-1} \times S^{n-1} \to S^{n-1}$ goes back to an old problem that asks which spheres S^{n-1} are *Hopf spaces* – topological spaces with a monoid structure. Of course, the map $S^{n-1} \times S^{n-1} \to S^{n-1}$ in question is just the monoid multiplication. (In addition to being of independent interest, this problem is also tied to the determination of the number of linearly independent vector fields on spheres, which Adams solved definitively in 1961 and laid out in [1].)

Now we describe the Hopf construction itself. Note that a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ is equivalent to a map $S^{n-1} \to Maps(S^{n-1}, S^{n-1})$; since rotations of S^{n-1} can be naturally extended to rotations of D^n that leave the origin fixed, our H-space multiplication μ can be written as an assignment $S^{n-1} \to Maps(D^n, D^n)$. Equivalently, there are extensions $\mu_+ : S^{n-1} \times D^n \to D^n$ and $\mu_- : D^n \times S^{n-1} \to D^n$. To describe a map out of S^{2n-1}, treat S^{2n-1} as the boundary of a $2n$-cell, and decompose as follows:

$$\begin{aligned} S^{2n-1} = \partial D^{2n} = \partial(D^n \times D^n) &= (\partial D^n \times D^n) \cup (D^n \times \partial D^n) \\ &= (S^{n-1} \times D^n) \cup (D^n \times S^{n-1}) \;. \end{aligned}$$

If we use μ_+ on the first piece and μ_- on the second piece, then we have a map $S^{2n-1} \to D^n \cup D^n$. But since μ_+ and μ_- are extensions of μ, they agree on $S^{n-1} \subset D^n$, and the hemispheres $D^n \cup D^n$ can be glued along S^{n-1} to form S^n.

Having found a way to generate elements of the elusive group $\pi_{2n-1}S^n$, we are now motivated to aim higher and look for elements of any $\pi_m S^n$. Happily, only a small modification of the above construction is necessary to produce useful results. Instead of the starting H-space multiplication $S^{n-1} \to Maps(D^n, D^n)$, let us begin with any element $f \in \pi_i O(n)$, which can be written as a map $S^i \times \mathbb{R}^n \to \mathbb{R}^n$. Then the decomposition of S^{2n-1} can be imitated to write

$$\begin{aligned} S^{n+i} = \partial D^{n+i+1} = \partial(D^{i+1} \times D^n) &= (\partial D^{i+1} \times D^n) \cup (D^{i+1} \times \partial D^n) \\ &= (S^i \times D^n) \cup (D^{i+1} \times S^{n-1}) \end{aligned}$$

Send the second piece to a point, and the first piece to D^n using the chosen map f. Gluing these together produces a map $S^{n+i} \to D^n \cup \{*\} = S^n$. One can show that this is a homomorphism when regarded as a map of homotopy groups $\pi_i O(n) \to \pi_{n+i} S^n$. This is the J-homomorphism.[1]

In the rest of this thesis, we will be working over $\mathbb{C}$ instead of $\mathbb{R}$, because it simplifies the arguments in several places. Going forward, the J-homomorphism will refer to the complex version

$$\pi_{2k-1} U(n) \to \pi_{2k-1+n} S^n$$

which is constructed in an analogous manner.

The beauty of this construction is that most of the groups $\pi_k U(n)$ are known:

$$\pi_k(U(n)) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \mathbb{Z} & \text{if } k \text{ is odd} \end{cases}$$

for large enough n (in a way that is described more precisely in Section 2.1). Thus, we know how to compute a subgroup $Im(J)$ of $\pi_{2k+n-1} S^n$. Understanding the image of the J-homomorphism will eventually allow us to prove that $Im(J)$ is a direct summand of one of the homotopy groups $\pi_{2k+n-1} S^n$. This, in turn, will be accomplished by constructing a sequence

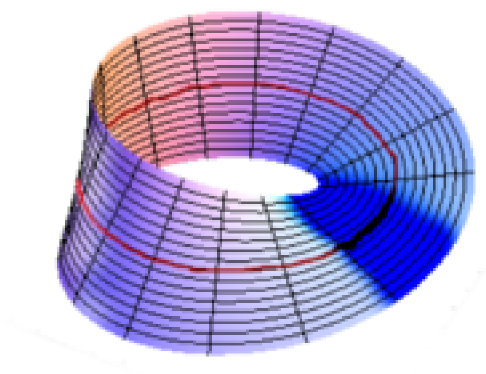
$$\pi_{2k-1} U(n) \xrightarrow{J} \pi_{2k+n-1} S^n \to \mathbb{Q}/\mathbb{Z} \ , \tag{1.1.1}$$

whose study will form the heart of this thesis. In the remainder of this chapter, we will introduce K-theory. In the second chapter, we lay out more background – constructions that will be invaluable to the rest of the discussion. In chapter 3, we will define the map $e : \pi_{2k+n-1} S^n \to \mathbb{Q}/\mathbb{Z}$ and determine its image. In chapter 4, we will study the composition (1.1.1), and will discuss the ideas necessary to obtain a splitting of the (stable) homotopy groups of spheres.

[1] The J-homomorphism was first defined by Whitehead (see [23]), and he called it H, perhaps to emphasize its connection to two other maps F and G which have faded from usage.

§1.2 Vector bundles

The Moebius strip is everyone's favorite vector bundle. Imagine yourself standing on the circle in the middle of the strip.



If all you could see was the shaded strip, it would look vaguely flat – practically indistinguishable from a strip on an ordinary, untwisted surface. We say that, *locally*, the Moebius band looks like a simple product space d(base circle) $\times D^1$. In general, vector bundles are spaces that locally look like a product: given a small enough neighborhood d(base space), the space looks like

$$\text{d(base space)} \times \mathbb{C}^n$$

for some n.

Definition 1.2.1. We say that $p : E \to B$ is a fiber bundle of dimension n with fiber F, if $p^{-1}(b) \cong F$ for every $b \in B$, and for every point $b \in B$ there is a neighborhood $U_b \subset B$ such that $p\big|_{U_b} \cong F \times B$. Usually we abuse notation and call E itself a vector bundle.

If the fiber is a vector space, then the bundle is called a vector bundle. If the fiber is a group G with a continuous action on the bundle, then it is called a G-principal bundle. The simplest fiber bundle is the Cartesian product $B \times F$ for any spaces B, F; this is called the *trivial bundle*. In general, fiber bundles are sometimes referred to as *twisted products*, vocabulary evocative of our motivating example, the Moebius strip.

Beyond the trivial bundle and the Moebius strip, one easy way of procuring vector bundles is via *clutching functions*. Suppose $X \to S^{2n}$ is a vector bundle of dimension d. Break the base space into two hemispheres $S^{2n} = D^n_+ \cup D^n_-$; as each piece is contractible, $X\big|_{D^n_+}$ and $X\big|_{D^n_-}$ are both trivial bundles isomorphic to $D^n \times \mathbb{C}^d$. There are subsets $S^{2n-1}_* \subset D^n_*$ of each hemisphere which are identified when S^{2n} is reassembled; X is entirely determined by the way in which $X\big|_{S^{2n-1}_+}$ and $X\big|_{S^{2n-1}_-}$ are glued. That is, it suffices to know, for every point in S^{2n-1}, a linear attaching map of the fibers $\mathbb{C}^d_+ \to \mathbb{C}^d_-$. In this case, we say that f is a *clutching function*, and because vector bundles over S^{2n} are in bijection with clutching functions, we may freely pass from a discussion of one to the other.

Though this particular construction does not generalize to all spaces, there is another way to describe vector bundles as *maps into* X. The space X in question is called the classifying space, and in our case, as long as the base space B is paracompact, there is a natural bijection

$$n\text{-dimensional vector bundles over } B \iff [B, G_n] \tag{1.2.1}$$

where G_n is the infinite-dimensional Grassmannian: the space of all n-dimensional linear subspaces of $\mathbb{C}^\infty$. In particular, every vector bundle $E \to B$ can be written as a pullback

$$\begin{array}{ccc} E & \longrightarrow & EG_k \\ \downarrow & & \downarrow \\ B & \longrightarrow & G_k \end{array} \tag{1.2.2}$$

where the total space EG_k is an infinite Stiefel manifold: the space of orthonormal k-dimensional frames in $\mathbb{C}^\infty$. The required map in $[B, G_k]$ is, of course, the bottom row of this diagram. Conversely, given a map in $[B, G_k]$, the associated vector bundle $E \to B$ is simply the pullback of the resulting diagram.

For example, $G_1(\mathbb{C}^\infty) = \mathbb{C}P^\infty$, so every line bundle is a pullback of the *universal line bundle* $H = \{(\ell, v) : v \text{ is on } \ell\} \subset \mathbb{C}P^\infty \times \mathbb{C}^\infty$.

Frame bundles

Sometimes it is more convenient to replace vector bundles with an equivalent category. There is a natural bijection

$$\text{Vector bundles of dimension } n \longleftrightarrow GL_n\text{-principal bundles} \tag{1.2.3}$$

In particular, given a vector bundle $p : E \to B$, for $b \in B$ let $F_b(E)$ be the set of bases of the vector space $p^{-1}(b)$; then the *frame bundle* $F(E)$ is defined to be the disjoint union $\coprod_{b \in B} F_b(E)$. There is an obvious action of GL_n on each $F_b(E)$ that takes a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ to $\{g \cdot \mathbf{e}_1, \cdots, g \cdot \mathbf{e}_n\}$ for $g \in GL_n$, and since the action is free and transitive, $F_b(E) \cong GL_n$. Thus $F(E)$ is a GL_n-principal bundle.

In the other direction, given a GL_n-bundle $E \to B$, construct the *associated vector bundle*

$$E \times_G \mathbb{C}^n = \{(e, v) \in E \times \mathbb{C}^n\}/(e, v) \sim (eg, g^{-1}v) \ .$$

There is a natural projection $E \times_G \mathbb{C}^n \to E$, with fiber isomorphic to $\mathbb{C}^n$; so this is indeed a vector bundle. One can show that these maps are actually inverses.

This helps us see the discussion of the Grassmannian in a new light: for any group G, we can attach a space BG, the *classifying space*, to the space of G-principal bundles, where:

- $[X, BG] \longleftrightarrow G$-principal bundles over X;
- $\Omega BG \simeq G$;
- if E is any weakly contractible space with a free G-action, then BG may be taken to be E/G.

Since GL_n is homotopy-equivalent to $U(n)$, vector bundles are equivalent to $U(n)$-bundles. So (1.2.1) is equivalent to the statement

$$BU(n) = G_n \ .$$

We will also need a similar construction.

Definition 1.2.2. Let H_d denote the set of homotopy equivalences $S^{d-1} \to S^{d-1}$.

Theorem 1.2.3. There is a space BH_d with $\Omega BH_d \simeq H_d$. This is the classifying space for S^{d-1} bundles up to fiber homotopy equivalence.

§1.3 Definition of K-theory

Consider the set of all vector bundles $p : E \to X$ over a space X. We would like to give this a group structure, so first we define the (direct) sum of vector bundles. Suppose $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are two vector bundles. Then define $E_1 \oplus E_2$ in the usual categorical way for defining products: as the pullback of

$$\begin{array}{ccc} E_1 \oplus E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ E_2 & \xrightarrow{p_2} & * \end{array}$$

Concretely, this is a subset of the product bundle $E_1 \times E_2 \to X \times X$ that ensures each pair has a canonical projection to X:

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \ : \ p_1(v_1) = p_2(v_2)\}.$$

One can show that this is a vector bundle (see [15], p. 10).

The idea is to use the direct sum operation to create a group structure $\widetilde{K}(X)$ out of the set of vector bundles over X. The obvious choice for "zero" in this group would be the trivial bundles. But this is initially problematic, because sometimes the direct sum of a trivial bundle with a non-trivial bundle can be trivial. For example, the tangent bundle to S^2, as a vector bundle over $\mathbb{R}$, is non-trivial (for example, by the Hairy Ball theorem), the normal bundle is trivial, and the sum is trivial. We would like to think that $T(S^2)$ is "almost trivial" because the addition of a trivial bundle makes it trivial. Vector bundles with this property are said to be *stably trivial*.

To salvage our original idea for a group, stably trivial bundles must be considered zero in the group structure. It turns out that this suffices as an equivalence relation. Let ε^n denote the trivial bundle of dimension n over X, and define an equivalence relation $\sim$, where $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^m \cong E_2 \oplus \varepsilon^n$. As desired, all stably trivial bundles are equivalent to trivial bundles. The existence of inverses can be shown using standard point-set topology (see [15], p.13).

The "unreduced" group $K(X)$ has a similar construction. As with reduced and unreduced cohomology, the most important property of $K(X)$ is that

$$K(X) \cong \widetilde{K}(X) \oplus K(x_0)$$

for some distinguished base point x_0.

In much the same way that we constructed the direct sum of vector bundles, we can also construct the tensor product. This makes $K(X)$ into a ring.

In general, $K(X)$ is not easy to compute. But the following facts can be proved using relatively elementary means (see chapter 2 of [15], or chapter 2 of [11]).

Theorem 1.3.1. The following hold:

- $K(\mathbb{C}P^n) \cong \mathbb{Z}[H]/(1-H)^{n+1}$
- $K(S^2) \cong K(\mathbb{C}P^1) \cong \mathbb{Z}[H]/(1-H)^2$, where H is the canonical line bundle over $\mathbb{C}P^1$
- (Bott periodicity) $K(\Sigma^2 X) = K(X)$, where Σ denotes the (reduced) suspension. Together with the fact that $\widetilde{K}(S^1) = 0$, induction then implies that

$$\widetilde{K}(S^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \text{ is even.} \end{cases} \tag{1.3.1}$$

Chapter 2

Some useful tools

In this chapter, we give a summary of the most important tools and concepts that will be needed in chapters 3 and 4.

§2.1 The stable category

One way to study $\pi_n(X)$ for some space X is to ask what happens in the limit as $n \to \infty$. The initial motivation to do came in 1937 with the proof of the Freudenthal suspension theorem.

Since the suspension functor Σ is left adjoint to the loop space functor Ω, there is an isomorphism $\mathrm{Hom}(\Sigma X, \Sigma X) \cong \mathrm{Hom}(X, \Omega\Sigma X)$, so the identity map $\Sigma X \to \Sigma X$ induces a map $X \to \Omega\Sigma X$, which in turn induces a map of homotopy groups $\pi_i X \to \pi_i \Omega\Sigma X = \pi_{i+1}\Sigma X$.

Theorem 2.1.1 (Freudenthal; see [14] p. 473)**.** Suppose X is an n-connected space.[1] The map $\pi_i X \to \pi_{i+1}\Sigma X$ given above is an isomorphism for $i \leq 2n$ and a surjection if $i = 2n+1$.

Corollary 2.1.2. $\pi_{i+n}S^n$ is independent of n, for $n \geq i+1$.

For n sufficiently large, we let π_i^s denote the "limit group" $\pi_{i+n}S^n$. The groups π_i^s are called the stable homotopy groups of spheres.

There have been many attempts to generalize the idea of stability in this sense. Instead of working in the category of CW complexes, we will be working in the *stable category* of CW spectra, as invented by Boardman and reformulated by Adams [8].

A CW spectrum is a collection of CW complexes E_n, with cellular maps $f_n : \Sigma E_n \to E_{n+1}$, where ΣE_n is given the structure of a CW complex by suspending each cell. As the suspension takes an i-dimensional cell c_i to an $(i+1)$-dimensional cell, the chain of maps $\cdots \to E_n \to E_{n+1} \to \dots$ induces a chain of maps of cells:

$$\dots \longrightarrow \underbrace{c_i}_{\substack{\text{dim. } i \\ \text{in } E_n}} \longrightarrow \underbrace{\Sigma c_i}_{\substack{\text{dim. } i+1 \\ \text{in } E_{n+1}}} \longrightarrow \underbrace{\Sigma^2 c_i}_{\substack{\text{dim. } i+2 \\ \text{in } E_{n+2}}} \longrightarrow \dots .$$

The direct limit $\lim_n \Sigma^n c_i$ is regarded as a *stable cell* of dimension $i - n$. Morphisms in this category are cellular maps of stable cells, up to homotopy.

[1] $\pi_1(X) = \cdots = \pi_n(X) = 0$

For the purposes of this thesis, we will not need to use the details of this construction, but will pass freely to the language of stability. Now we record the stable versions of many of the objects we have seen in the previous chapter.

Theorem 2.1.3. The homotopy group $\pi_i U(n)$ is independent of n, as long as $n > \frac{i}{2}$.

We will notate the stable group as $\pi_i U$.

Proof. I claim that there is a fiber bundle

$$U(n) \to U(n+1) \to S^{2n+1}$$

$U(n+1)$ is the group of rotations of $S^{2n+1} \subset \mathbb{C}^{n+1}$, where the stabilizer of a point v (a unit vector) consists of the rotations that preserve the n-dimensional orthogonal complement of v; this is isomorphic to $U(n)$. Now construct the long exact homotopy sequence of a fibration:

$$\cdots \to \pi_i U(n) \to \pi_i U(n+1) \to \pi_i S^{2n+1} \to \pi_{i-1} U(n) \to \pi_{i-1} U(n+1) \to \cdots$$

Since $\pi_i S^{2n+1} = 0$ if $i < 2n+1$, this gives isomorphisms $\pi_i U(n) \to \pi_i U(n+1)$ in the desired range. □

The J-homomorphism can be written as a *stable* map

$$\pi_{2k-1} U \to \pi^s_{2k-1} \ .$$

Furthermore, the inclusions

$$H_1 \subset H_2 \subset H_3 \subset \cdots$$

and

$$U_1 \subset U_2 \subset U_3 \subset \cdots \ ,$$

give stable classifying spaces $BU = \bigcup BU(n)$ and $BH = \bigcup BH_n$. Just as $BU(n)$ classified n-dimensional vector bundles, BU classifies the "virtual" sums of vector bundles in $\widetilde{K}(X)$: there is a correspondence

$$\widetilde{K}(X) \longleftrightarrow [X, BU] \ .$$

In particular, $\widetilde{K}(S^n) \cong \pi_n BU$.

§2.2 Adams operations

Like the addition and tensor multiplication operations associated to $K(X)$, Adams operations provide another form of structure. In an arbitrary cohomology theory, cohomology operations are maps $h(X) \to h(Y)$ which are natural. For example, in ordinary cohomology we have the Steenrod squares $Sq^i : H^n(X, \mathbb{Z}/2) \to H^{n+i}(X, \mathbb{Z}/2)$. In K-theory, we have Adams operations $\psi^k : K(X) \to K(X)$ for every k.

Adams operations were first defined in [1] in order to count the number of linearly independent vector fields on S^n. Adams originally defined ψ^k as a *virtual representation*, an object consisting of GL_n-representations over each $x \in X$ (up to equivalence), in much the same way that $\xi \in K(X)$ consists of vector spaces over each $x \in X$ (up to equivalence).

By the correspondence laid out in Section 1.2, instead of working directly with elements of $K(X)$, we may work with frame bundles (up to equivalence). We can define the exterior product of frame bundles fiberwise, where

$$\lambda^1 : \{v_1, \ldots, v_n\} \mapsto \{v_1, \ldots, v_n\},$$

$$\lambda_2 : \{v_1,\ \ldots,\ v_n\} \mapsto \{v_1 \wedge v_2,\ v_1 \wedge v_3,\ v_1 \wedge v_4,\ \ldots,\ v_2 \wedge v_3,\ v_2 \wedge v_4,\ \ldots\},$$

and in general,

$$\lambda^k\{v_1, \cdots, v_n\} = \left\{\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}\right\}_{\text{sets of } k \text{ distinct indices } i_1, \cdots, i_k}.$$

Furthermore, we impose a condition to make this antisymmetric:

$$v_1 \wedge \cdots \wedge v_n = \text{sign}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$$

where σ is a permutation. Such a map on bases induces a map on frame bundles $F(E) \to \lambda^k F(E)$, and hence a map on K-theory. (Note that λ^k maps an n-dimensional vector bundle to an $\binom{n}{k}$-dimensional vector bundle.)

By the splitting principle, we should first be concerned with defining the Adams operations on sums $L_1 \oplus \cdots \oplus L_\nu$ of line bundles; we would like ψ^k to preserve the sum decomposition, where the dimension is determined by k: in particular, $\psi^k : L_1 \oplus \ldots \oplus L_\nu \mapsto L_1^k \oplus \ldots \oplus L_\nu^k$. In this case, exterior powers on vector bundles look much the same as exterior powers on vector spaces:

$$\lambda^k(L_1 \oplus \cdots \oplus L_n) \quad = \sum_{\substack{\text{sets of } k \text{ distinct} \\ \text{indices } i_1, \cdots, i_k}} L_{i_1} L_{i_2} \cdot \cdots \cdot L_{i_n}\ .$$

The goal is to find a polynomial combination of the vector bundles $\lambda^i(L_1 \oplus \cdots \oplus L_\nu)$ that equals $L_1^k \oplus \cdots \oplus L_\nu^k$.

Consider the *symmetric polynomials*, that is, those polynomials $f(x_1, \dots, x_n)$ that are invariant under permutations of the x_i. The *elementary symmetric polynomials* s_k are polynomials that look like the exterior powers above:

$$s_1 = x_1 + \cdots + x_n \qquad\qquad s_2 = \sum_{i \neq j \leq n} x_i x_j$$

$$s_k = \sum_{\substack{i_1 \dots i_k \\ distinct}} x_{i_1} \dots x_{i_k}$$

It is a fact going back to Newton that every symmetric polynomial is generated by a polynomial combination of s_k. In particular, there is some polynomial Q_k^N such that

$$x_1^N + \cdots + x_k^N = Q_k^N(s_1, \dots, s_k)$$

This suggests we define

$$\psi^k(E) = Q^k_{\dim E}(\lambda^1(E), \cdots, \lambda^{\dim E}(E))$$

and so, by construction, we have

$$\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k \oplus \cdots \oplus L_n^k \ .$$

For reference, we reproduce the statement of the following theorems from [11] and [15].

Theorem 2.2.1 ([15], Thm. 2.20, 2.21; [11], Prop. 3.2.1, 3.2.2)**.** The Adams operations, as defined above, enjoy the following properties:

(1) $\psi^k f^* = f^* \psi^k$ for all maps $f : X \to Y$

(2) $\psi^k(L) = L^k$ if L is a line bundle

(3) $\psi^k \circ \psi^\ell = \psi^{k\ell}$

(4) $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$

(5) $\psi^k(xy) = \psi^k(x)\psi^k(y)$

(6) $\psi^k : \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$ is given by $x \mapsto k^n \cdot x$

§2.3 Localization

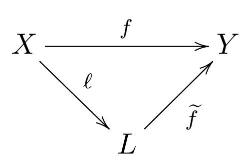
When investigating an unknown finite abelian group like π_i^s, it is often easier to proceed "one prime at a time": that is, to focus on the p-torsion by throwing away (inverting) all other primes. This is the process of localization. We may wish $\pi_*(X)$[2] or $H_*(X)$ were local, but as an approximation, we hope to approximate X by a derived space X_p such that $\pi_*(X_p)$ or $H_*(X_p)$ are local. It turns out that there is no distinction between localizing homotopy and localizing homology.

Proposition 2.3.1. $\pi_*(X)$ is local at p if and only if $H_*(X)$ is local at p.

Proof. Theorem 2.2 of [21]. □

Then the *localization* of a space X is defined by a universal property as the "local space closest to X":

Definition 2.3.2. Suppose there exists a space L and a map $X \xrightarrow{\ell} L$ such that, for all maps $f : X \to Y$ into a local space Y, there is a unique map $\widetilde{f}$ such that



commutes. Then L is the *localization* of X.

The example that seems tailor-made for this construction is the Eilenberg-Maclane space. Recall that $K(A, n)$ is the space where $\pi_n(K(A, n)) = A$ and all other homotopy groups are zero. If A_p is the localization of A at p, it is rather clear that $K(A_p, n)$ is the localization of $K(A, n)$ at p.

Of course, Eilenberg-Maclane spaces are important for their algebra more than their geometry. When we say we are *localizing a space*, the implication is that there is something about the geometry of the construction that "looks like" localization. The sphere provides a good illustration of this. First, let us recall what precisely is meant by localization in the algebraic sense. Inverting a prime q is equivalent to taking the direct limit

$$\mathbb{Z} \xrightarrow{q} \mathbb{Z} \xrightarrow{q^2} \mathbb{Z} \xrightarrow{q^3} \mathbb{Z} \to \dots \ .$$

[2] $\pi_*(X)$ is not a ring, but by "local" we mean that it is a $\mathbb{Z}_p$-module.

(To see this, write the limit as $\coprod_n \mathbb{Z}/\sim\ \cong\ \coprod_n \frac{1}{q^n}\mathbb{Z}\ /\sim$, where the equivalence relation simply ensures that $q^e \cdot \left(\frac{1}{q^n}y\right) \sim q^{e-n}y$, making it isomorphic to $\left\{\frac{a}{p^n}\ :\ 0 \leq n\right\}$ as desired.) Localizing $\mathbb{Z}$ at p amounts to inverting *all* products of primes $q \neq p$, and this is similarly equivalent to a direct limit construction. In particular, let (a_n) be an enumeration of all the numbers not divisible by p, and let $A_n = \prod_{i=1}^{n} a_i$. Then localizing $\mathbb{Z}$ at p amounts to taking the direct limit

$$\mathbb{Z} \xrightarrow{A_1} \mathbb{Z} \xrightarrow{A_2} \mathbb{Z} \xrightarrow{A_3} \mathbb{Z} \to \dots \quad . \tag{2.3.1}$$

Having transformed the algebraic process of localization into a process that makes sense in any category, we may apply this to **Top**. In particular, for every n there is a degree map $S^k \xrightarrow{n} S^k$, so we may define the localization of S^k to be the (homotopy) direct limit of the sequence

$$S^k \xrightarrow{A_1} S^k \xrightarrow{A_2} S^k \xrightarrow{A_3} S^k \to \dots$$

More precisely, we define the localization to be the infinite mapping telescope (that is, a chain of mapping cylinders). (To see the connection to direct limits, take the simple case where there is only one map $A \xrightarrow{f} B$. Then the direct limit is the disjoint union $A \sqcup B$ modulo the relation that $a \sim f(a)$. The mapping cylinder is the same construction, except that A is replaced by $A \times [0,1]$, to which it is homotopy equivalent. The *homotopy direct limit* formalizes this idea for infinite families.)

Note that this does indeed localize homology: we need only check $H_*(S^k)$ in degree k, and since $H_k(S^k) = \mathbb{Z}$, this is exactly the construction in (2.3.1).

Unfortunately, not every space X comes with a convenient map $X \xrightarrow{n} X$. But we can still localize CW complexes using the above construction, along with induction on the n-skeleton. Details can be found in the proof of Theorem 2.2 of [21].

§2.4 Thom complexes

Basic properties

The Thom complex is a standard construction that is of interest here because it provides a useful description of the mapping cone C_f of a map $f \in \pi_{2(n+k)-1}S^{2n}$.

Definition 2.4.1 (Thom complex)**.** If $E \xrightarrow{\xi} B$ is a vector bundle, then define $D(E)$ to be the *disc bundle*, obtained by restricting each fiber of E to $D^q \subset \mathbb{C}^q$. Define the *sphere* bundle $S(E)$ by restricting the fibers to $S^{q-1} \subset \mathbb{C}^q$. Finally, define the Thom complex $\mathrm{Thom}(B, \xi) = D(E)/S(E)$. (That is, copies of S^{q-1} over every point $b \in B$ are collected and collapsed to a point.)

Up to homotopy equivalence we may quotient out each fiber by $D^q - \{0\}$ instead of S^{q-1}, so if $E' \subset E$ is the complex obtained by removing 0 from every fiber, the Thom complex can be written as the relative pair (E, E').

The Thom isomorphism holds for any multiplicative generalized cohomology theory, but we will use it here in the context of K-theory.

Theorem 2.4.2 (Thom isomorphism theorem)**.** Given a vector bundle $E \xrightarrow{\xi} B$ of dimension q, there is an element $U_\xi \in \mathrm{Thom}(B, \xi)$ whose restriction to each fiber D^q/S^{q-1} is a generator in K-theory. Furthermore, the map

$$K^i(B) \xrightarrow{T} \widetilde{K}^{i+q}(T(E)) \ , \qquad\qquad x \mapsto x \otimes U_\xi$$

is an isomorphism. The element U_ξ is called the Thom class.

Now we give a useful description of an arbitrary Thom complex as a mapping cylinder. Choose an arbitrary vector bundle ξ over S^{r+1}, which we shall identify by its clutching function $\varphi : S^r \to U(n)$.

Lemma 2.4.3.

$$\mathrm{Thom}(S^{r+1}, \xi) = S^q \cup_{J\varphi} C(S^{q+r})$$

for some q.

Since $C(S^{q+r}) = D^{q+r+1}$, in order to represent $\mathrm{Thom}(S^{r+1}, \xi)$ as a mapping cylinder, we need to first represent it as a union $S^q \cup D^{q+r+1}$ for some q, find a copy of $S^{q+r} \subset D^{q+r+1}$ and a map $S^{q+r} \to S^q$, and show that $\mathrm{Thom}(S^{r+1}, \xi)$ is formed by gluing the union along this map.

The vector bundle ξ is the union of two trivial vector bundles, the restrictions to the upper and lower hemispheres of S^r, glued by the clutching function φ. The fibers are copies of $\mathbb{C}^q$, but up to homotopy we may assume the fibers are D^q instead. Since the hemispheres are contractible, collapse S^r_- to a point p. This lets us write ξ again as a union of trivial vector bundles $D^q \cup (D^q \times D^{r+1})$, with gluing still given by φ. Now we find the copy of $S^{q+r} \simeq \partial(D^q \times D^{r+1}) \simeq (D^q \times \partial D^{r+1}) \cup (\partial D \times D^{r+1}) \subset D^q \times D^{r+1}$: the first piece $\mathbb{C}^q \times \partial D^{r+1}$ is the boundary circle (see Figure 2.1), along with its fiber, to be identified with the fiber of p; the second piece $\partial\mathbb{C}^q \times D^{r+1}$ contains, for every point in D^{r+1}, the subset S^{q-1} of the fiber that will be identified to a point in the Thom construction.

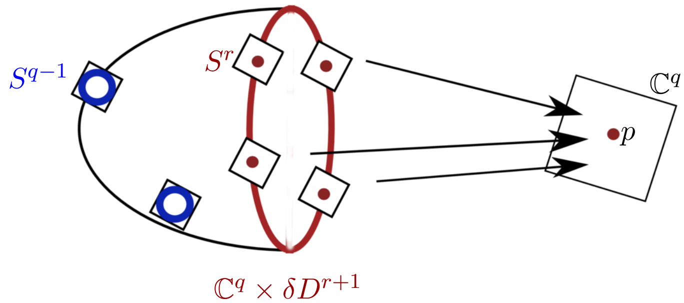


Figure 2.1:

To form the Thom complex, first glue the two parts of ξ together using φ, and then collapse $\mathbb{C}^q \times \partial D^{r+1}$, as well as a copy of S^{q-1} over p, to a point. But this is precisely what $J\varphi : S^{q+r}$ does: it applies φ to : $\partial(\mathbb{C}^q \times D^{r+1}) \subset S^{q+r}$ and collapses the rest of S^{q+r}.

The cannibalistic class

In [3], Adams defined a characteristic class which he called the cannibalistic class:

$$\chi_\ell(\xi) := \frac{1}{\ell^d} \cdot T^{-1}(\psi^\ell U_\xi) \in K(X)$$

where ξ is a bundle of dimension d, as illustrated by the diagram

$$\begin{array}{ccc} K^i(B) & \xrightarrow{T} & \widetilde{K}^{i+d}(T(E)) \\ \downarrow{\scriptstyle \chi_\ell} & & \downarrow{\scriptstyle \psi_\ell} \\ K^i(B) & \xleftarrow[T^{-1}]{} & \widetilde{K}^{i+d}(T(E)) \end{array} \quad \ni \quad \begin{array}{ccc} 1 & \xrightarrow{T} & U_\xi \\ \downarrow & & \downarrow{\scriptstyle \psi^\ell} \\ \ell^d \cdot \chi_\ell(\xi) & \xleftarrow[T^{-1}]{} & \psi^\ell U_\xi \end{array} \tag{2.4.1}$$

(Since T is just multiplication by U_ξ, we often write T^{-1} as division, so $X/U_\xi := T^{-1}X$.) This has some nice properties, as recorded below.

Proposition 2.4.4.

$$\chi_\ell(\xi_1 \oplus \xi_2) = \chi_\ell(\xi_1)\, \chi_\ell(\xi_2)$$

The following example justifies the choice of normalization.

Proposition 2.4.5.

$$\chi_\ell(\mathbb{C}^d) = 1$$

where $\mathbb{C}^d$ represents the d-dimensional trivial bundle over a point.

Proof. Here $D(\mathbb{C}^d) = D^d$, and $S(\mathbb{C}^d) = S^{2d-1}$, so the quotient is S^{2d}. (Recall that S^{2X} has $2X$ real dimensions, and thus X complex dimensions, which explains the extra factor of 2 here.) The Thom isomorphism thus maps

$$\mathbb{Z} = K(*) \to \widetilde{K}(S^{2d}) = \mathbb{Z}$$

and the Thom class is the generator of $\widetilde{K}(S^{2d})$. Therefore, we can write $T : 1 \mapsto 1$ in this case. The Adams operation ψ^ℓ acts on $\widetilde{K}(S^{2d})$ by multiplication by ℓ^d. In sum, in this case diagram (2.4.1) looks like

$$\begin{array}{ccc} 1 & \longrightarrow & d \\ \downarrow & & \downarrow \\ \ell^d & \longleftarrow & \ell^d \end{array}$$

□

Lemma 2.4.6. Let H be a line bundle. Then

$$\chi_\ell(H^t) = \frac{1 - H^{\ell \cdot t}}{\ell^k(1 - H^t)} \ .$$

We need the following proposition:

Proposition 2.4.7. If L is the canonical line bundle over $\mathbb{C}P^\infty$, then $\mathrm{Thom}(\mathbb{C}P^\infty, L) \cong \mathbb{C}P^\infty$.

Proof. If L_n is the canonical line bundle over $\mathbb{C}P^n$, then I first claim that $L_n \cong \mathbb{C}P^{n+1}$. If $(\mathbf{x}, \mathbf{a}) := \big([x_0, \cdots, x_n], (a_0, \cdots, a_n)\big)$ is a typical element of $L_n \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$, then construct the isomorphism to send $(\mathbf{x}, \mathbf{a}) \mapsto [a_i, x_0, \cdots, x_n]$, where i is the first index such that $x_i \neq 0$. (That is, we are normalizing $\mathbf{x}$ by setting the first nontrivial element to 1, and using this to measure how $\mathbf{a}$ is a scalar multiple of $\mathbf{x}$.) The inverse map $L \to \mathbb{C}P^{n+1}$ is just the projection. Either by taking the limit, or by using the same construction, one can show that $L \cong \mathbb{C}P^\infty$.

In order to form the Thom complex, take the quotient by $S(L)$, the subset of L containing $(\mathbf{x}, \mathbf{a})$ where $|\mathbf{a}| = 1$. However, by projecting $(\mathbf{x}, \mathbf{a}) \mapsto \mathbf{a}$ we see that $S(L) \cong S^\infty$. It is a well-known fact that $S^\infty \simeq *$, so

$$\mathbb{C}P^\infty \cong L \cong D(L) \cong D(L)/S(L) = \mathrm{Thom}(\mathbb{C}P^\infty, L) \ .$$

□

Proof of Lemma 2.4.6. We have

$$U_H = 1 - H \in K(\mathbb{C}P^\infty)$$

and

$$\chi_k(H) = \frac{1}{k^n} \cdot \frac{\psi^k U_H}{U_H} = \frac{1}{k^n} \cdot \frac{\psi^k(1-H)}{1-H} = \frac{1}{k^n} \cdot \frac{1-H^k}{1-H} \ .$$

Furthermore, using the naturality principle of Adams operations, $U_{H^t} = 1 - H^t$ and so

$$\chi_\ell(H^t) = \frac{1}{\ell^k} \cdot \frac{\psi^\ell U_{H^t}}{U_{H^t}} = \frac{1}{\ell^k} \cdot \frac{1-H^{\ell\cdot t}}{1-H^t} \ .$$

□

Chapter 3

The e-invariant

In this chapter, we will calculate a lower bound for the order of $Im(J)$. The bound can be stated in terms of Bernoulli numbers, which have a long history and many uses in number theory. I will give a brief introduction here, following [16], so the reader can appreciate their unexpected arrival in the realm of topology.

§3.1 Bernoulli numbers

The Bernoulli numbers B_k were discovered by Jacob Bernoulli, who was originally trying to find a formula for sums $1^k + 2^k + \cdots + n^k$. In particular, he defined the numbers B_k such that

$$1^m + 2^m + \cdots + (n-1)^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k} \ .$$

All of the odd B_k are zero, except for B_1.

The Bernoulli numbers have an appealing generating function

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!} \ .$$

and have deep connections to number theory, which can perhaps be glimpsed in the beautiful formulation involving the Riemann zeta function:

$$B_{2k} = -2(2k)!(2\pi i)^{-2k} \zeta(2k) n^{-2k} \ .$$

There is a concise description of the prime factorization of their denominators, which is a consequence of the following theorem.

Theorem 3.1.1 (Clausen – von Staudt)**.**

$$(-1)^k B_{2k} \equiv \sum_{\substack{p \ prime \\ p-1 | 2k}} \frac{1}{p} \pmod{\mathbb{Z}} \ .$$

The denominators of the B_{2k} are thus divisible only by those primes p such that $p-1 \mid 2k$. If ℓ is such a prime, then multiplying both sides of the expression in the theorem yields

$$\mathrm{ord}_\ell(\ell \cdot (-1)^k B_{2k}) = \mathrm{ord}_\ell \Bigg(\sum_{\substack{p \neq \ell \\ p-1 | 2k}} \frac{\ell}{p} + 1 + \ell \cdot N \Bigg) = 0,$$

which gives the following corollary.

Corollary 3.1.2. If D_{2k} is the (reduced) denominator of B_{2k}, then for a prime p we have

$$\operatorname{ord}_p D_{2k} = \begin{cases} 1 & \text{if } p-1 \mid 2k \\ 0 & \text{otherwise.} \end{cases}$$

§3.2 Defining the e-invariant

Our story begins with an element $f \in \pi_{2(k+n)-1} S^{2n}$, which can be interpreted as the attaching map of a cell $e_{2(k+n)} \cup S^{2n}$. Construct the Puppe sequence associated with f:

$$S^{2(k+n)-1} \xrightarrow{f} S^{2n} \longrightarrow \underbrace{S^{2(k+n)-1} \cup_f CS^{2n}}_{\text{mapping cone } C_f} \longrightarrow \Sigma S^{2(k+n)-1} \xrightarrow{\Sigma f} \Sigma S^{2n} \longrightarrow \cdots$$

Any Puppe sequence can be made into a long exact sequence in $\widetilde{K}$-theory (see [15], p.52), which in this case gives

$$\widetilde{K}(S^{2(k+n)-1}) \longleftarrow \widetilde{K}(S^{2n}) \longleftarrow \widetilde{K}(C_f) \longleftarrow \widetilde{K}(\Sigma S^{2(k+n)-1}) \longleftarrow \widetilde{K}(\Sigma S^{2n}) \longleftarrow \cdots \tag{3.2.1}$$

By (1.3.1), we have found a short exact sequence

$$0 \longleftarrow \mathbb{Z} \longleftarrow \widetilde{K}(C_f) \longleftarrow \mathbb{Z} \longleftarrow 0 \tag{3.2.2}$$

In order to understand $f \in \pi_{2(k+n)-1} S^{2n}$, we will study the mapping cone C_f, and in particular, its K-theory. The sequence (3.2.2) shows that $\widetilde{K}(C_f) \cong \mathbb{Z} \oplus \mathbb{Z}$ for every such f, but there is still more information to be mined in the exact configuration of the maps in this sequence. Each sequence of the form (3.2.2) represents an element in $\operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z})$. Roughly speaking, the e-invariant assigns to f the element of $\operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z})$ corresponding to (3.2.2) However, we have yet to add a crucial restriction: (3.2.2) respects the Adams operations of Section 2.2.

We will now work in the category of finitely generated abelian groups equipped with Adams operations ψ^ℓ for every ℓ (that is, endomorphisms satisfying the properties of Theorem 2.2.1).

For ease of notation, let $\mathbb{Z}(t)$ denote $\mathbb{Z}$ with the Adams operations $\psi^\ell : x \mapsto \ell^t \cdot x$.

Using (3.2.1) to determine which Adams operations act on the groups, (3.2.2) can be written in this new category as

$$0 \longleftarrow \mathbb{Z}(2n) \longleftarrow \widetilde{K}(C_f) \longleftarrow \mathbb{Z}(2(k+n)) \longleftarrow 0 \ . \tag{3.2.3}$$

Now pass to the stable groups: that is, we may take n above to be as large as desired. In fact, most of the time it will simplify notation, and not omit important details, if we ignore n entirely; thus we will write

$$0 \longleftarrow \mathbb{Z}(0) \longleftarrow \widetilde{K}(C_f) \longleftarrow \mathbb{Z}(2k) \longleftarrow 0 \ . \tag{3.2.4}$$

instead of (3.2.3). We are now ready to define the e-invariant.

Definition 3.2.1. For $f \in \pi^s_{2k-1}$, define $e(f)$ to be the element of $\mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(2k))$ corresponding to (3.2.4).

It turns out that this is a homomorphism; see, for example, [15] p. 103.

§3.3 The image of e

The goal of this section is to show that the image of e is a finite group, and to obtain an upper bound on its size.

Lemma 3.3.1. There is an isomorphism

$$\mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(2k)) \cong \mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}/\mathbb{Z}(2k)) \ .$$

Proof. Starting with the short exact sequence

$$0 \to \mathbb{Z}(2k) \to \mathbb{Q}(2k) \to \mathbb{Z}/\mathbb{Q}(2k) \to 0$$

we may construct the Hom-Ext long exact sequence:

$$\begin{aligned} 0 \to &\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Z}(2k)) \to \mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}(2k)) \to \mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}/\mathbb{Z}(2k)) \\ &\xrightarrow{\Phi} \mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(2k)) \to \mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Q}(2k)) \to \mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Q}/\mathbb{Z}(2k)) \to 0 \ . \end{aligned} \tag{3.3.1}$$

The desired statement now follows from Claims 3.3.2 and 3.3.3 below. □

Claim 3.3.2.

$$\mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Q}(2k)) = 0$$

Proof. Given any sequence $\mathbb{Q}(2k) \xrightarrow{\eta} A \xrightarrow{\varepsilon} \mathbb{Z}(0)$, we construct a splitting $\mathbb{Z}(0) \to A$. Let $\widetilde{a} \in \mathbb{Z}(0)$ be any preimage of $1 \in \mathbb{Z}(0)$, and ℓ any number; write

$$a := \widetilde{a} - \frac{\psi^\ell \widetilde{a} - \widetilde{a}}{\ell^{2k} - 1} = \frac{\ell^{2k}\widetilde{a} - \psi^\ell \widetilde{a}}{\ell^{2k} - 1} \ .$$

There is a homomorphism $h : \mathbb{Z} \to A$ generated by $1 \mapsto a$; we will show that this is compatible with the Adams operations, and in fact forms a splitting of ε. To show that $\psi \circ h(n) = h \circ \psi(n)$, note that $\psi \circ h(n) = \psi(n \cdot a) = n \cdot \psi(a)$ and $h \circ \psi(n) = h(n) = n \cdot a$, so it suffices to show that $\psi(a) = a$. Since by hypothesis $\varepsilon(\widetilde{a}) = 1 \in \mathbb{Z}(0)$ and ψ^ℓ is trivial on the image of ε, we have

$$\varepsilon(\psi^\ell \widetilde{a} - \widetilde{a}) = \varepsilon(\psi^\ell \widetilde{a}) - \varepsilon(\widetilde{a}) = \psi^\ell(\varepsilon(\widetilde{a})) - \varepsilon(\widetilde{a}) = 0 \ .$$

By definition of our short exact sequence, this implies that $\ell^{2k}\widetilde{a} - \widetilde{a} \in \mathbb{Q}(2k)$, and so $\psi^\ell(\psi^\ell \widetilde{a} - \widetilde{a}) = \ell^{2k}(\psi^\ell \widetilde{a} - \widetilde{a})$. Now calculate

$$\psi^\ell a = \psi^\ell \widetilde{a} - \frac{\ell^{2k}(\psi^\ell \widetilde{a} - \widetilde{a})}{\ell^{2k} - 1} = \frac{-\psi^\ell \widetilde{a} + \ell^{2k}\widetilde{a}}{\ell^{2k} - 1} = a \ .$$

To check that it is a splitting, use the facts above that $\varepsilon(\widetilde{a}) = 1$ and $\varepsilon(\ell^{2k}\widetilde{a} - \widetilde{a}) = 0$ to compute

$$\varepsilon(h(n)) = n \cdot \varepsilon(a) = n \cdot \left(\varepsilon(\widetilde{a}) - \frac{\varepsilon(\psi^\ell \widetilde{a} - \widetilde{a})}{\ell^{2k} - 1} \right) = n$$

as desired.

□

Claim 3.3.3.

$$\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}(2k)) = 0$$

Proof. The group $\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}(2k))$ contains maps $g : \mathbb{Z} \to \mathbb{Q}$ such that $\ell^{2k} g(x) = g(\psi^\ell x) = \psi^\ell g(x) = g(x)$. So $g(x) = 0$ for all x, and this Hom group is zero. □

In order to obtain a first approximation of the size of $Im(e)$, we will calculate the size of the target space, and by Lemma 3.3.1, it suffices to calculate the size of $\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}/\mathbb{Z}(2k))$. Since the connecting homomorphism in (3.3.1) is invertible, given a sequence

$$0 \to \mathbb{Z}(2k) \to C_{Jf} \to \mathbb{Z}(0) \to 0,$$

we may construct a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(2k) & \longrightarrow & C_{Jf} & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{g} & & \\ 0 & \longrightarrow & \mathbb{Z}(2k) & \longrightarrow & \mathbb{Q}(2k) & \longrightarrow & \mathbb{Q}/\mathbb{Z}(2k) & \longrightarrow & 0 \end{array} \tag{3.3.2}$$

where g is the homomorphism associated to the sequence. Each g is determined by $g(1)$, and in order to find the order of $\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}/\mathbb{Z}(2k))$, it suffices to the order of the group generated by all possible $g(1) \in \mathbb{Q}/\mathbb{Z}$; if we write $g(1) = \frac{a}{b}$ in lowest terms, the goal is to find the possible denominators b.

We must now momentarily abandon the shorthand $\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Q}/\mathbb{Z}(2k))$ and write this group as an explicit limit $\mathrm{Hom}(\mathbb{Z}(N), \mathbb{Q}/\mathbb{Z}(2k+N))$ for N sufficiently large. Since g respects the Adams operations, we have

$$\ell^{2k+N}\frac{a}{b} \equiv \ell^N \frac{a}{b} \pmod{\mathbb{Z}}$$

or equivalently, $b \mid a \cdot \ell^N(\ell^{2k} - 1)$ for all ℓ. If we assume that the fraction $\frac{a}{b}$ is in lowest terms, then $(a, b) = 1$ and it is equivalent to ask that $b \mid \ell^N(\ell^{2k} - 1)$. Thus, to find the prime decomposition of all possible $g(1)$, it suffices to find, for every prime p with $(p, \ell) = 1$, the power j such that

$$\ell^{2k} \equiv 1 \pmod{p^j} \tag{3.3.3}$$

for all ℓ.

We may write the multiplicative group $(\mathbb{Z}/p^j)^\times$ as an additive $\mathbb{Z}$-module, so that (3.3.3) becomes

$$2k \cdot (\mathbb{Z}/p^{j-1})^\times = 0.$$

In order for $2k$ to annihilate all of $(\mathbb{Z}/p^{j-1})^\times$, it must be zero in that group. By Lemma 3.3.6, if p is odd this happens only when $p^{j-1} \mid 2k$ and $p - 1 \mid 2k$. Using similar reasoning for $p = 2$, and invoking Lemma 3.3.7 we have proven the following theorem.

Theorem 3.3.4. Let m_{2k} be the largest number that divides $\ell^{N(\ell)}(\ell^{2k} - 1)$ for all ℓ, given sufficiently large $N(\ell)$. Then

(1) the e-invariant is a map

$$e : \pi^s_{2k-1} \to \mathbb{Z}/m(2k) \ ;$$

(2) m_{2k} has a prime decomposition

$$m_{2k} = 2^a p_1^{j_1} \cdot \dots \cdot p_m^{j_m}$$

where j_i is the largest number such that $\frac{p_i - 1}{2} \mid k$, and $p_i^{j_i - 1} \mid k$; the exponent a is the largest integer such that $2^{a-3} \mid k$.

Corollary 3.3.5. We have that

$$m_{2k} = 2 \ \times \ \text{denominator of } \ \frac{B_{2k}}{2k} \ .$$

Proof. Theorem 3.1.1 (Clausen – von Staudt theorem). □

Now we present proofs of the number-theoretic facts referenced above.

Lemma 3.3.6. If p is an odd prime, then

$$(\mathbb{Z}/p^j)^\times \cong \mathbb{Z}/(p-1) \oplus \mathbb{Z}/p^{j-1} \; .$$

Proof. There is an exact sequence

$$0 \to (1 + p \cdot \mathbb{Z}/p^j)^\times \to (\mathbb{Z}/p^j)^\times \to (\mathbb{Z}/p)^\times \to 0 \; .$$

In fact, there is a splitting $(\mathbb{Z}/p)^\times \to (\mathbb{Z}/p^j)^\times$ given by $x \mapsto x^{p^{j-1}}$. Since $(1 + p \cdot \mathbb{Z}/p^j)^\times$ contains the elements $1 + np$, it can be identified with $\mathbb{Z}/p^{j-1}$; the term on the right can be canonically identified with $\mathbb{Z}/(p-1)$. Since a split exact sequence is equivalent to a direct sum decomposition, the lemma follows. □

Lemma 3.3.7.

$$(\mathbb{Z}/2^j)^\times \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^{j-2}$$

Proof. The aim is to find a generator of $\mathbb{Z}/2^{j-2}$. Since $(\mathbb{Z}/2^j)^\times$ is the group of odd elements, it has order 2^{j-1}. We will first show that every element has order at most 2^{j-1}; this shows that $(\mathbb{Z}/2^j)^\times \neq \mathbb{Z}/2^{j-1}$. Then we will find elements of order exactly 2^{j-1}; this shows that $(\mathbb{Z}/2^j)^\times$ contains a subgroup isomorphic to $\mathbb{Z}/2^{j-2}$, from which the lemma follows by the classification of finite abelian groups.

Let $a \in (\mathbb{Z}/2^j)^\times$. We have $a^2 = a + 2 \cdot N$ for some N by Fermat's little theorem. Squaring both sides, we have $a^4 = a^2 + 4(aN + N^2)$. But $aN + N^2$ is even: this is clear if N is even, and otherwise aN and N^2 are both odd because a is odd. Squaring both sides repeatedly, we have

$$\begin{aligned}
a^4 &= a^2 + 8\left(\frac{aN + N^2}{2}\right) =: a^2 + 8N_4 \\
a^8 &= a^4 + 16(a^2 N_4 + 4N_4^2) =: a^4 + 16N_8 \\
&\vdots \\
a^{2^n} &= a^{2^{n-1}} + 2^{n+1}(a^{2^{n-2}} N_{2^{n-1}} + 2^{n-1} N_{2^{n-1}}^2) =: a^{2^{n-1}} + 2^{n+1} N_{2^n}
\end{aligned} \tag{3.3.4}$$

Since $(\mathbb{Z}/2^j)^\times$ has order 2^{j-1}, we know that $a^{2^{j-1}} \equiv 1 \pmod{2^j}$. But by the above, we have

$$a^{2^{j-2}} \equiv a^{2^{j-1}} \equiv 1 \pmod{2^j}$$

so a has order at most 2^{j-2}.

Now we revisit this argument more carefully to find an element of order exactly 2^{j-2}. Using (3.3.4) with $n = j - 2$, we have

$$1 \equiv a^{2^{j-2}} = a^{2^{j-3}} + 2^{j-1}N \pmod{2^j} \ .$$

If $a^{2^{j-3}} \equiv 1 \pmod{2^j}$, then $2^{j-1}N \equiv 0$. So it suffices to show that every N_i is odd, given some $a \in (\mathbb{Z}/2^j)^\times$.

Suppose $a \equiv 3 \pmod 8$ (for example, $a = 3$). Then $a^2 - a \equiv 6 \pmod 8$; if $2N = a^2 - a$ as above, then $N \equiv 3 \pmod 4$. Furthermore, $aN + N^2 \equiv 3 \cdot 3 + 3^2 \equiv 2 \pmod 4$, which implies N_4 is odd. Using induction and (3.3.4), we find that every N_{2^i} is odd, as desired.

□

Chapter 4
A splitting for π^s_{2k-1}

Using the map e discussed in the last chapter, we can make the composition

$$\pi_{2k-1}U \xrightarrow{J} \pi^s_{2k-1} \xrightarrow{e} \mathbb{Q}/\mathbb{Z}$$

In this chapter, we will almost show that the inclusion $Im(J) \hookrightarrow \pi^s_{2k-1}$ produces a splitting. More specifically, in the diagram

$$\begin{array}{ccc} Im(J) & \xrightarrow{\varphi} & \mathbb{Z}/m_{2k} \\ {\scriptstyle i}\downarrow & \nearrow {\scriptstyle e} & \\ \pi^s_k & & \end{array} \tag{4.0.1}$$

we want the composition marked φ to be an isomorphism; this would show that $\varphi^{-1} \circ e$ is the required splitting. Injectivity will be proven in Section 4.3, by describing the image of the J-homomorphism in terms of a new functor $J(X)$. In section 4.1, we present slightly simplified methods that take us a factor of 2 away from proving surjectivity: instead of proving that $|\mathbb{Z}/m_{2k}|$ divides $|Im(J)|$, we show that $|\mathbb{Z}/m_{2k}|$ divides $2|Im(J)|$. This is a result of working over $\mathbb{C}$. The extra factor of 2 can be recovered with slightly more complication, by using $\mathbb{R}$-vector bundles. Details can be found in Adams' original papers [2]-[5].

§4.1 (Almost) surjectivity

In the previous chapter, we showed that the e-invariant, an element of $\mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(2k))$, can be represented as the image of g in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(2k) & \longrightarrow & C_{Jf} & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & \mathbb{Z}(2k) & \xrightarrow{i} & \mathbb{Q}(2k) & \longrightarrow & \mathbb{Q}/\mathbb{Z}(2k) & \longrightarrow & 0 \end{array}$$

By 2.4.3, C_{Jf} is a Thom complex, so let $\widetilde{b} \in C_{Jf}$ be the Thom class. This projects to the generator b of $\mathbb{Z}(0)$. Choose a generator $a \in \mathbb{Z}(2k)$ and identify this with its image in C_{Jf}. Since $a, \widetilde{b}$ are generators of C_{Jf}, we can express $\psi^\ell(\widetilde{b}) = ma + n\widetilde{b}$, and since $\widetilde{b} \mapsto b$ under the projection to $\mathbb{Z}(0)$, we have $n = 1$. Since $a \in M$ comes from an element of $\mathbb{Z}(2k)$, ψ^ℓ acts on it by multiplication by ℓ^{2k}. So ψ^ℓ acts on M by the matrix $\begin{pmatrix} \ell^{2k} & m \\ 0 & 1 \end{pmatrix}$. There is a one-dimensional ψ^ℓ-invariant subspace; this is the kernel of $\psi^\ell - Id = \begin{pmatrix} \ell^{2k}-1 & m \\ 0 & 0 \end{pmatrix}$. The kernel of this operator is generated by $t := (\ell^{2k}-1)\widetilde{b} - m \cdot a$. Since $\zeta(t)$ is ψ^ℓ-invariant, it must be zero in $\mathbb{Q}(2k)$. The e-invariant is the image of b in $\mathbb{Q}/\mathbb{Z}(2k)$; so it suffices to find $\zeta(\widetilde{b})$ modulo $\mathbb{Z}$. We have:

$$\begin{aligned} 0 &= \zeta((\ell^{2k}-1)\widetilde{b} - m \cdot a) \\ (\ell^{2k}-1)\zeta(\widetilde{b}) &= m\zeta(a) \\ \zeta(\widetilde{b}) &= \frac{m}{\ell^{2k}-1}\zeta(a) \end{aligned}$$

Since i is an inclusion and a comes from the generator of $\mathbb{Z}(2k)$, its image in $\mathbb{Q}(2k)$ is just 1. So

$$e(f) = \frac{m}{\ell^{2k}-1} \quad . \tag{4.1.1}$$

From above,

$$m \cdot a = \psi\widetilde{b} - \widetilde{b} \ .$$

Use the inverse of the Thom isomorphism, which takes $m \cdot a \mapsto \psi\widetilde{b}/\widetilde{b} - 1 = \chi_\ell(\widetilde{b}) - 1$, The computation of $\chi_\ell(\widetilde{b})$ is long and has been deferred to the appendix. Using Lemma A.1, we have

$$e(f) = \frac{\chi_\ell(\widetilde{b}) - 1}{\ell^{2k}-1} = \frac{B_{2k}}{2k} \ .$$

By Corollary 3.3.5, $e(f)$ generates a group of order $\frac{1}{2}m_{2k}$. If the image of $e \circ J$ was m_{2k} instead, then we would have shown that this composition was surjective.

§4.2 The groups $\widetilde{J}(X)$

It remains to show that the map $\varphi : Im(J) \to \mathbb{Z}/m_{2k}$ in (4.0.1) is an injection. This follows from the theorem below, whose proof we will sketch in the present section.

Theorem 4.2.1. $Im(J)$ is a cyclic group of order dividing m_{2k}.

Following [2], we will define a quotient $\widetilde{J}(X)$ of $\widetilde{K}(X)$, with the property that $Im(J) \subset \pi^s_{2k-1}$ is isomorphic to $\widetilde{J}(S^{2k})$.

Every vector bundle has an *associated sphere bundle*, obtained by taking the quotient $D^n \to S^n$ on each fiber. Now consider associated sphere bundles up to *fiber homotopy equivalence*: the natural way to define homotopy equivalence on bundles. More precisely, if $p : E \to B$

and $p' : E' \to B$ are fiber bundles, then we say that E and F are fiber homotopy equivalent if there is a homotopy equivalence $f : E \to E'$ whose restriction to B is the identity.

Now define the functor $J(X)$ to be the quotient $K(X)/\sim$, where $\xi \sim \eta$ if the sphere bundles associated to ξ and η are fiber homotopy equivalent. Similarly, define $\widetilde{J}(X) = \widetilde{K}(X)/\sim$. This definition is useful for our purposes because of the following lemma.

Lemma 4.2.2. $\widetilde{J}(S^{2k})$ can be identified with the image of the stable J-homomorphism $\pi_{2k-1}U \xrightarrow{J} \pi^s_{2k-1}$.

Proof. This is proven in [9], so only a sketch will be presented here.

$\widetilde{J}(X)$ is the group of virtual sphere bundles that are restrictions of virtual vector bundles. Before passing to the stable category, we can describe spherical fibration restrictions of n-dimensional vector bundles as the image of the map:

$$\rho_n : \{n\text{-dimensional } \mathbb{C}\text{-vector bundles over } X\} \to \{S^{2n-1} \text{ bundles over } X\}$$

which can also be written $\rho_n : [X, BU(n)] \to [X, BH(2n)]$. One hopes that $\widetilde{J}(X)$ is actually the limit of such maps, and this fact is proven in [12].

Now we give an overview of how to make ρ_n look like the J-homomorphism. On the one hand, as discussed above, the image of ρ_n can be identified with $\widetilde{J}(X)$. On the other hand, plugging in $X = S^{2k}$ produces a map $\pi_{2k}BU(n) \to \pi_{2k}BH(2k)$ that is almost the J-homomorphism. In order to relate homotopy groups of $H(n)$ to homotopy groups of spheres, we need the following lemma.

Lemma 4.2.3. We have that

$$\pi_{r-1}H(n) = \pi_{n+r-2}(S^{n-1}) \ .$$

Proof. Details can be found in [9]. The idea is to consider the subset $H_n^+ \subset H_n$ of degree 1 maps and construct a fibration $H_n^+ \to S^{n-1}$ with fiber $\Omega^{n-1}S^{n-1}$, given by evaluation at a fixed base point. The lemma follows by manipulation of the long exact homotopy sequence of the fibration. □

Then we have

$$[S^{2k}, BU(n)] = \pi_{2k}BU(n) = \pi_{2k-1}\Omega BU(n) = \pi_{2k-1}U(n)$$
$$[S^{2k}, BH(n)] = \pi_{2k}BH(n) = \pi_{2k-1}\Omega BH(n) = \pi_{2k-1}H(n) = \pi_{n+2k-2}S^{n-1} \ .$$

After passing to the stable category, this becomes a map $\rho : \pi_{2k-1}U \to \pi_{2k-1}^s$.

By retracing this construction, it can be shown that ρ is, in fact, the J-homomorphism. □

§4.3 A special case of the Adams conjecture

Because $Im(J)$ is a finite cyclic group, by the previous section we may write $Im(J) = \widetilde{J}(S^{2k}) = \mathbb{Z}/n_{2k}$. Theorem 4.2.1 states that $n_{2k} \mid m_{2k}$. Since m_{2k} is the largest number dividing all $\ell^{N(\ell)}(\ell^k - 1)$ (for sufficiently large choices of $N(\ell)$), to prove $e \circ J$ is injective it suffices to show that n_{2k} also divides all $\ell^{N(\ell)}(\ell^k - 1)$ for sufficiently large $N(\ell)$. This is a special case of the "Adams conjecture," named such because it originally appeared as a conjecture in Adams' paper [2]. It is, however, no longer a "conjecture": proofs may be found in [22] and [20].

Theorem 4.3.1 (Adams conjecture)**.** For every ℓ, there exists $N(\ell)$ sufficiently large such that $\ell^{N(\ell)}(\ell^k - 1)$ goes to zero in $\widetilde{J}(X)$. Equivalently, $\psi^\ell - 1$ goes to zero under the natural map $\widetilde{K}(X) \to \widetilde{J}(X) \to \widetilde{J}(X) \otimes \mathbb{Z}[\frac{1}{\ell}]$.

Both proofs cited above go beyond the scope of my work here, but I will sketch a quick proof for the case of interest.

It suffices to show the statement is true in every localization $\widetilde{J}(X)_p$ (this can be realized as a quotient of $K(X)_p$).

Proposition 4.3.2. If S is a multiplicative subset of a ring R, then the localization M_S has no S-torsion.

Proof. Consider the canonical map $M \to M_S = M \otimes R_S$ given by $m \mapsto m \otimes 1$. Then the S-torsion of M is sent to zero under this map:

$$m \otimes 1 = m \otimes (p^n \cdot \tfrac{1}{p^n}) = (p^n \cdot m) \otimes \frac{1}{p^n} = 0 \qquad \text{if } m \in T(S).$$

□

If $\widetilde{J}(X)$ is localized at $p \mid \ell$, then the result is obvious: the arbitrarily large power ℓ^N kills the p-torsion, and localizing kills all torsion other than p-torsion. Thus it suffices to consider localizing at a prime p with $(p, \ell) = 1$, and so the statement reduces to

$$(\psi^\ell - 1)y = 0 \text{ in } \widetilde{J}(X)_p \quad \text{for } (p, \ell) = 1,$$

or equivalently, that ψ^ℓ is the identity map on $\widetilde{J}(X)_p$.

Let $E \to X$ be the associated sphere bundle of a vector bundle. In order to show that ψ^ℓ is the identity, it suffices to show that the induced map f on fibers is always an isomorphism after localizing.

$$\begin{array}{ccc} S^n_p & \xrightarrow{f} & S^n_p \\ \downarrow & & \downarrow \\ E_p & \xrightarrow{\psi^\ell} & \psi^\ell E_p \\ & \searrow \quad \swarrow & \\ & X_p & \end{array} \tag{4.3.1}$$

(One can show that localizing $J(X)$ is equivalent to localizing all the spaces in the diagram.)

Claim 4.3.3. The Adams conjecture is true for line bundles.

Proof. Let $L \to X$ be a line bundle; the associated sphere bundle can be denoted by $L \times_{U(1)} S^1$. We know what Adams operations do to line bundles: $\psi^\ell : L \to L^k$. So ψ^ℓ induces a map $\psi^\ell : L \times_{U(1)} S^1 \to L \times_{U(1)} S^1$, where $U(1)$ acts on S^1 by $x \mapsto x$ on the first copy of $L \times_{U(1)} S^1$, and $x \mapsto x^\ell$ on the second copy. This is summarized in the diagram.

$$\begin{array}{ccc} L \times_{U(1)} S^1 & \xrightarrow{\ell} & L \times_{U(1)} S^1 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

□

Claim 4.3.4. The Adams conjecture is true for sums $\bigoplus_i L_i$ of line bundles.

Claim 4.3.5. If E is a pullback in the following diagram

$$\begin{array}{ccc} E & \longrightarrow & \bigoplus_i L_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

and $f^* : J(Y) \to J(X)$ is a monomorphism, then the Adams conjecture is true for E.

This is not quite as good as it seems at first: the splitting principle guarantees similar conditions, but with a monomorphism in K-theory, not J-theory. However, in the special case of spheres, we can take advantage of the smash product to fulfill this condition.

Proposition 4.3.6. If

$$s : S^2 \times \cdots \times S^2 \longrightarrow S^2 \wedge \cdots \wedge S^2 = S^{2n}$$

is the smash product, then $s^* : J(S^{2n}) \to J(S^2 \times \cdots \times S^2)$ is a monomorphism.

In fact, this proposition suffices to prove the Adams conjecture for spheres. By Bott periodicity, $K(S^2 \times \cdots \times S^2) \cong K(S^2) \otimes \cdots \otimes K(S^2)$, which gives a splitting $J(S^2 \times \cdots \times S^2) \cong J(S^2) \otimes \cdots \otimes J(S^2)$. As each $K(S^2)$ is generated by line bundles, so is $J(S^2)$, so the conditions on Claim 4.3.5 are satisfied.

Lemma 4.3.7. Let Z be an H-space, and assume that $\pi_0 Z$ is a group. Then we have that

$$[X \times Y, Z] = [*, Z] \times [X, Z]_* \times [Y, Z]_* \times [X \wedge Y, Z]_* \ .$$

This gives an injection $[X \wedge Y, Z] \hookrightarrow [X \times Y, Z]$.

Proof. First note that $[X, Z]$ is a group, for any space Z: if $f, g : X \to Z$ then define

$$f * g : X \xrightarrow{(f,g)} Z \times Z \xrightarrow{mult} Z \ .$$

There is a sequence

$$[X \wedge Y, Z]_* \to [X \times Y, Z]_* \xrightarrow{\alpha} [X \vee Y, Z]_*$$

where the first map is induced by the quotient $X \times Y \to X \wedge Y$, and the second map α is induced by the inclusion $X \vee Y \hookrightarrow X \times Y$. Since $[X \vee Y, Z] = [X, Z]_* \times [Y, Z]_*$, I claim that

$$\beta : [X, Z]_* \times [Y, Z]_* \to [X \times Y, Z]_* \qquad \text{where} \quad \beta(f, g) : X \times Y \xrightarrow{(f,g)} Z \times Z \xrightarrow{mult} Z$$

is a splitting for $\alpha : [X \times Y, Z]_* \to [X, Z]_* \times [Y, Z]_*$, written as $\alpha : f \mapsto (f(-, *)\ ,\ f(*, -))$. Indeed, given $X \to Z$ and $Y \to Z$, the components of $\alpha \circ \beta$ are given by $X \times * \to Z \times * \to Z$ and $* \times Y \to * \times Z \to Z$, which are, by definition, the original maps. □

Proof (Proposition 4.3.6). Recall we had a map $BU \to BH$ that classifies ρ : vector bundles $\to$ sphere bundles. It turns out that BH satisfies the conditions of Lemma 4.3.7. Since $S^{2n} = S^2 \wedge \cdots \wedge S^2$, using $Z = BH$ and induction we obtain an injection $[S^{2n}, BH] \hookrightarrow [S^2 \times \cdots \times S^2, BH]$. If we let $KSph$ denote the functor represented by BH (so $KSph(X)$ is the group of spherical fibrations over X), then $KSph(S^{2n}) \hookrightarrow KSph(S^{2n} \times \cdots \times S^{2n})$. Since $J(X) \subset KSph(X)$, the proposition follows.

□

Appendix A

Cannibalistic class computation

Let $b \in K(S^{2n})$ be a generator. By the splitting principle, $b \to S^{2n}$ splits as a sum of line bundles:

$$\begin{array}{ccc} H^t & \longrightarrow & b \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \longrightarrow & S^{2n} \end{array}$$

Since $\mathrm{Thom}(\mathbb{C}P^n, H) = 1 - H$, by naturality we have $\mathrm{Thom}(\mathbb{C}P^n, H^t) = (1-H)^t$, and therefore $\mathrm{Thom}(S^{2n}, b)$ can be identified with $(1-H)^n$.

Lemma A.1. Let X be the Thom class of S^{2n}, and H be the canonical line bundle. Then

$$\chi_\ell(X) = 1 + \frac{B_n}{n}(\ell^k - 1)(1-H)^n \ .$$

Proof. Write

$$\begin{aligned} X &= (-1)^n (H_1 - 1) \otimes (H_2 - 1) \otimes \cdots \otimes (H_n - 1) \\ &= \underbrace{(H_1 + \cdots + H_n)}_{\sigma_{1,n}(H_1,\cdots,H_n)} - \cdots + (-1)^n \underbrace{H_1 H_2 \cdots H_n}_{\sigma_{n,n}(H_1,\cdots,H_n)} \end{aligned}$$

where $\sigma_{k,n}$ is the elementary symmetric polynomial of degree k in n variables $H_1, \cdots, H_n$. Let $y_i = H_i - 1$, and note that $y_i^2 = 0$ because $y_i \in K(S^2) = K(\mathbb{C}P^n) = \mathbb{Z}[H]/(H-1)^2$. Since $e^t = 1 + t + \cdots$ we can write $H_i = 1 + y_i = e^{y_i}$. Then

$$\chi_k(H_i) = \frac{1 - H_i^k}{1 - H_i} = \frac{1 - e^{k y_i}}{1 - e^{y_i}}$$

Now

$$\chi_k(X) = \frac{\chi_k(\sigma_{1,n})\,\chi_k(\sigma_{3,n})\cdots}{\chi_k(\sigma_{2,n})\,\chi_k(\sigma_{4,n})\cdots}$$

where $\sigma_{i,n}$ are elementary symmetric polynomials on the n variables $y_1, \cdots, y_n$. We can write each $\sigma_{i,n}$ as a sum of its constituent degree-i monomials, and use Proposition 2.4.4:

$$\chi_k(X) = \frac{\prod_{\substack{\text{degree 1} \\ \text{monomials } M}} \chi_k(M) \prod_{\substack{\text{degree 3} \\ \text{monomials } M}} \chi_k(M) \cdots}{\prod_{\substack{\text{degree 2} \\ \text{monomials } M}} \chi_k(M) \prod_{\substack{\text{degree 4} \\ \text{monomials } M}} \chi_k(M) \cdots} \ . \tag{A.1.1}$$

By naturality,

$$\chi_\ell(H_1 H_2) = \frac{1 - e^{\ell(y_1 + y_2)}}{1 - e^{y_1 + y_2}} \tag{A.1.2}$$

and so on. This suggests the definition

$$g(y) := \frac{1 - e^{ky}}{1 - e^{y}} \ .$$

So if a monomial M looks like H_1H_2, then we need notation for writing it additively (with the extra factor of t that was introduced above): define M' in this case to be $y_1 + y_2$. So (A.1.1) can be written additively as

$$\chi_\ell(X) = \frac{\prod_{\substack{M' \text{ with odd} \\ \# \text{ of terms}}} g(M')}{\prod_{\substack{M' \text{ with even} \\ \# \text{ of terms}}} g(M')} \ . \tag{A.1.3}$$

By (A.1.2) and (A.1.3), the constant term of any $g(M')$ (as a power series in $y_1, \cdots, y_n$) is 1, so the constant term of $\chi_\ell(X)$ is 1. So we can write $\chi_\ell(X) = 1 + \mu \cdot F(y_1, \cdots, y_n)$. We will now find μ. Introduce a new variable t, and substitute $y_i = tx_i$; we now regard g as a function of x_i and t. To turn the products in (A.1.1) into sums, take logarithmic derivatives (in terms of t). Let $f = \frac{g'}{g}$ and write the logarithmic derivative of (A.1.3) as

$$\begin{aligned}
\frac{d}{dt}\log(1 + \mu F(tx_1, \cdots, tx_n)) &= \frac{\deg F \cdot \mu F'(tx_1, \cdots, tx_n)}{1 + \mu F(tx_1, \cdots, tx_n)} \\
&= (\deg F \cdot \mu \cdot F'(tx_1, \cdots, tx_n))(1 + \text{ higher terms}) \\
&= f(tx_1) + \cdots + f(tx_n) - f(tx_1 + tx_2) + \cdots + (-1)^n f(tx_1 + \cdots + tx_n) \ .
\end{aligned}$$

Now we obtain another expression for f:

$$\begin{aligned}
f := \frac{g'}{g} &= \frac{-kxe^{kxt}}{1 - e^{kxt}} - \frac{-xe^{xt}}{1 - e^{xt}} = \frac{kx}{1 - e^{-kxt}} - \frac{x}{1 - e^{-xt}} \\
&= \frac{1}{t}\left[\sum \frac{B_j}{j!} x^j t^j - \sum \frac{B_j}{j!} x^j t^j\right] = \frac{1}{t}\left[\sum \frac{B_j}{j!}(\ell^k - 1) x^j t^j\right]
\end{aligned} \tag{A.1.4}$$

and plug this into (A.1.4) to obtain

$$\begin{aligned}
\frac{\mathcal{A}'}{\mathcal{A}} &:= \deg F \cdot \mu \cdot F'(tx_1, \cdots, tx_n) + \cdots \qquad\qquad (A.1.5) \\
&= \frac{1}{t}\sum \frac{B_j}{j!}(\ell^k - 1)\Big[(tx_1)^j + \cdots + (tx_n)^j - (tx_1 + tx_2)^j + \cdots + (-1)^n (tx_1 + \cdots + tx_n)^j\Big].
\end{aligned}$$

This motivates one more piece of notation:

$$\begin{aligned}
h_{j,1} &:= x_1^j \\
h_{j,2} &:= x_1^j + x_2^j - (x_1 + x_2)^j \\
h_{j,3} &:= x_1^j + x_2^j + x_3^j - (x_1 + x_2)^j - (x_1 + x_3)^j - (x_2 + x_3)^j + (x_1 + x_2 + x_3)^j
\end{aligned}$$

$$\cdots h_{j,k} := \sum (-1)^{\nu+1}(x_{i_1} + \cdots + x_{i_\nu})^j \ .$$

So we can write

$$\frac{\mathcal{A}'}{\mathcal{A}} = \frac{1}{t}\sum \frac{B_j}{j!}(\ell^k - 1)h_{j,n}. \tag{A.1.6}$$

Claim A.2. There is a recurrence relation

$$h_{j,k}(x_1, \cdots, x_k) = h_{j,k-1}(x_1, \cdots, x_{k-1} + x_k) - h_{j,k-1}(x_1, \cdots, x_{k-1}) - h_{j,k-1}(x_1, \cdots, x_k)$$

Proof. To avoid even more notational clutter, consider only $h_{j,6}(a, b, c, d, y, z)$. We will break up the constituent terms of each $h_{j,*}$ such that one term in $h_{j,6}$ is uniquely accounted for by a sum of terms on the right.

First, focus only on terms (like $(a+b)^j$ or $(b+c+d)^j$) on the left that do not contain y or z. Each such term appears once in each of the three terms on the left; for example, we might have $(a + b)^j - (a + b)^j - (a + b)^j$. This contributes one copy of $-(a + b)^j$, where the sign has been switched from $h_{j,5}$.

Now focus on the terms in $h_{j,6}$ (like $(a + y)^j$) containing y but not z. These correspond one-to-one with the terms of $h_{j,5}(a, b, c, d, y)$ that contain y. Similarly, terms in $h_{j,6}$ containing z but not y are in 1-1 correspondence with the terms of $h_{j,5}(a, b, c, d, z)$ containing z. The terms on the left still not accounted for are those containing both y and z, like $(a+b+c+y+z)^j$. These are uniquely given by the terms of $h_{j,5}(a, b, c, d, y + z)$ that involve the last slot. □

Recall that $x_i^2 = 0$ for all i.

Claim A.3. If $j > k$, then $h_{j,k} = 0$.

Proof. The expansion of any term like $(x_1 + \cdots + x_k)^j = x_1^j + x_1^{j-1}x_2 + \cdots$ is a homogeneous polynomial in degree j. If $j > k$, then some variable x_i must occur more than once in each monomial. □

Claim A.4. If $j < k$, then $h_{j,k} = 0$.

Proof. By the recurrence relation given above, it suffices to prove this for $h_{j,j+1}$. By the reasoning in the Claim A.3, it suffices to consider only the terms $(x_{i_1} + \cdots + x_{i_\nu})^j$ where $\nu \geq j$. This includes one term $(x_1 + \cdots + x_{j+1})^j$, and $j + 1$ terms $(x_2 + \cdots + x_{j+1})^j$,

$(x_1 + x_3 + \cdots + x_{j+1})$, ..., $(x_1 + \cdots + x_{j+1})^j$. Again using the fact that $x_r^2 = 0$ for all r, we have:

$$(x_1 + \cdots + \widehat{x_i} + \cdots + x_{j+1})^j = a x_1 x_2 \cdots \widehat{x_i} \cdots x_{j+1} \tag{A.4.1}$$

where a counts the number of ways to choose some x_r (with $r \neq i$) for every factor $(x_1 + \cdots + \widehat{x_i} + \cdots + x_{j+1})$ such that no x_r is chosen twice. That is, $a = j!$. Similarly, if $x_r^2 = 0$ then

$$\begin{aligned}(x_1 + \cdots + x_{j+1})^j \\ = b_1(x_2 + \cdots + x_{j+1}) + \cdots + b_i(x_1 + \cdots + \widehat{x_i} + \cdots + x_{j+1}) + \cdots + b_{j+1}(x_1 + \cdots + x_j)\end{aligned} \tag{A.4.2}$$

where $b_i = j!$ for the same reason. Since (A.4.1) and (A.4.2) have different signs in $h_{j,j+1}$, they cancel out and $h_{j,j+1} = 0$ under the equivalence relation $x_i^2 = 0$. □

Claim A.5. If $j = k$, then $h_{j,k} = j! x_1 \cdots x_j$.

Proof. By the reasoning in Claim A.3, we only need to consider the term $(x_1 + \cdots + x_j)^j$ in $h_{j,j}$. By the reasoning in Claim A.4,

$$(x_1 + \cdots + x_j)^j = j! x_1 \ldots x_j$$

if $x_i^2 = 0$ for all i. □

Finally, going back to (A.1.6), we see that every term disappears except the one containing $h_{n,n}$. So

$$\mathcal{A} = \frac{1}{t}\frac{B_n}{n!}(\ell^k - 1) n! \cdot x_1 \cdots x_n t^n \ . \tag{A.5.1}$$

From (A.1.5), we have

$$\frac{\mathcal{A}'}{\mathcal{A}} = 1 \cdot (\deg F \cdot t^{\deg F - 1} \cdot \mu F(tx_1, \cdots, tx_n)) + \text{ higher terms} = \frac{1}{t}\frac{B_n}{n!}(\ell^k - 1) \cdot n! x_n \cdots x_n t^n$$

(The higher terms are zero, because they each must contain some factor of x_i^2.) This shows that $\deg F = n$, and so $n\mu = B_n$. Since H_i all represent the same universal line bundle, we obtain the statement of the lemma.

□

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