10 The Impact of the Number Type on the Solution of Multiplication and Division Problems: Further Considerations

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INTRODUCTION

Fischbein, Deri, Nello, and Marino (1985) argue that students' conceptions of and performance on multiplication and division application word problems (hereafter multiplicative problems) are unconsciously derived from primitive intuitive models that "correspond to features of human mental behavior that are primary, natural and basic" (p. 15). They suggest their theory to account for conceptions, such as multiplication makes bigger and division makes smaller, identified in previous studies (e.g., Bell, Fischbein, and Greer, 1984; Vergnaud, 1983; Bell, Swan, and Taylor, 1981). These conceptions are in accord with the operations of multiplication and division in the whole number domain but incongruent to these operations in the rational number domain; thus, they block the way to correctly solve many multiplicative problems whose quantities are decimals or fractions.

In our recent effort to better understand the multiplicative conceptual field, and in particular the transition phase from the additive structure to the multiplicative structure, we probed into this question of incongruity. We found that many questions and concerns are still open and need further investigation. Among these we addressed the following: (1) the impact of the number type on the solution of multiplicative problems; (2) the impact of the textual structure on problem interpretation; and (3) solution models used by teachers to solve multiplicative problems. To address some of these questions empirically, we developed an instrument that controls a wide range of confounding variables known to be influential on subjects' performance on multiplicative problems and used it with inservice and preservice teachers. In this chapter we report on an investigation of aspect (1); the investigations on the other two aspects will be reported separately.

THEORETICAL BACKGROUND

We start with a summary of Fischbein et al.'s models for multiplication and division and the evidence reported by some studies for their impact. Then we address, in this order, questions concerning the instruments used to investigate these models, a specific constraint these models impose on the numerical aspect of the quantities representing the multipliers and divisors, and the relative robustness of the intuitive rules associated with these intuitive models.

According to Fischbein et al. (1985), the model associated with multiplication problems is *repeated addition*. Under this model the roles played by the quantities multiplied are asymmetrical (Greer, 1985). One of the factor quantities, called the *multiplier*, is conceived of as the number of equivalent collections, while the other quantity, called the *multiplicand*, is conceived of as the size of each collection. These conceptions apparently lead subjects to intuit the rule that multipliers must be whole numbers, which, in turn, results in another rule: the product must be larger than the multiplicand, or multiplication makes bigger (Bell et al., 1984; Hart, 1984; Bell et al., 1981). For division, Fischbein et al. suggested two intuitive models, one is associated with equal sharing, or *partitive division* problems, the other with measurement, or *quotitive division* problems. In the partitive division model, an object or collection of objects is partitioned into a number of equivalent fragments or subcollections. The size of the object or the number of objects is represented by the dividend, the number of the equivalent fragments or subcollections is represented by the divisor, and the size of each equivalent fragment or subcollection is represented by the quotient. As a result, certain rules are intuited and become associated with this model: (1) the divisor must be a whole number, (2) the divisor must be smaller than the dividend,

TABLE 10.1. INTUITIVE RULES ASSOCIATED WITH FISCHBEIN'S THREE MODELS

Operation	*Intuitive rules*
Multiplication	1. Multiplier must be a whole number 2. Multiplication makes bigger
Partitive division	1. Divisor must be a whole number 2. Divisor must be smaller than dividend 3. Division makes smaller
Quotitive division	1. Divisor must be smaller than dividend

and (3) the quotient must be smaller than the dividend, or division makes smaller, which is derived from the previous two rules. The quotitive division model is associated with division problems in which it is required to find how many times a given quantity is contained in another quantity. The only constraint imposed by this model is that divisors must be smaller than dividends. Table 10.1 summaries these intuitive rules.

Fischbein et al. (1985) looked for substantiation of their theory in an investigation with fifth, seventh, and ninth graders. Other researchers tested this theory with different populations. For example, Greaber, Tirosh, and Glover (1989) replicated Fischbein et al.'s study with preservice elementary teachers and further investigated the similarity between their conceptions and children's conceptions of multiplication and division. Mangan (1986) addressed Fischbein et al.'s theory in a study that systematically controlled the contextual and the number type variables, with a wide age range of children and adults: primary and secondary school students, continuing education students, university students, and student teachers.

In general, these studies and others (see Greer, 1988) support Fischbein et al.'s theory; that is, they are consistent with the argument that subjects's solution of multiplicative problems is affected by their intuitive models about multiplication and division and by the numerical constraints imposed by them. The following are some of the main results of these studies.

The impact of the repeated addition model was substantiated by the finding that the solution of multiplication word problems is affected by the type of multiplier. It was shown that problems with a whole-number multiplier are significantly easier than problems with a decimal multiplier larger than 1; and the latter were easier than those with a (positive) decimal multiplier smaller than 1 (e.g., Mangan, 1986; De Corte, Verschaffel, and Van Coillie, 1988). Also, consistent with the repeated addition model is the finding that no significant difference in problem difficulty was found whether the multiplicand was represented as a whole number, a decimal greater than 1, or a decimal smaller than one (Luke, 1988).

The impact of the division models also was substantiated. For example, Mangan (1986) showed that problem quantities consistent with the partitive constraints (e.g., 25 ÷ 8 and 26.85 ÷ 9) yielded the highest level of performance; quantities that violate the constraint that the divisor must be a whole number (e.g., 11.44 ÷ 4.51, 32 ÷ 5.69, 5.87 ÷ 0.44, and 8 ÷ 0.77) resulted in a decrease in the percentage of correct responses; and quantities that violate the constraint that the divisor must be smaller than dividend (7 ÷ 23 and 0.38 ÷ 0.89) resulted in the lowest level of performance. Similarly, in quotitive division quantities that conform to the quotitive constraint (25 ÷ 8, 26.85 ÷ 9, 11.44 ÷ 4.51, 32 ÷ 5.69, 5.87 ÷ 0.44, and 8 ÷ 0.77) yielded a higher performance than those that violate it (7 ÷ 23 and 0.38 ÷ 0.89).

In an analysis of these and other studies, we made several observations. The first, which has been a concern to other researchers as well (e.g., Nesher, 1988; Goldin, 1986; Lester and Kloosterman, 1985), is that the problems used in these studies controlled only for the number variable, leaving uncontrolled all or some other important variables such as context, text, and syntax, which are known to be important factors in the research on additive word problems. The instrument we developed for this study (described in the next section) was designed to take into consideration this concern.

The second observation from these studies is that it seems that decimal multipliers greater than 1 are treated by subjects like whole numbers: Subjects seem to have little difficulty solving multiplication problems with a non-whole-number multiplier greater than 1 (compared to the difficulty they have with problems with a decimal smaller than 1) despite the fact that such problems violate the same intuitive rule, that the multiplier must be a whole number. Our question was, What is the conceptual basis for this phenomenon? Fischbein et al. noticed this phenomenon and suggested the notion of an *absorption effect* to explain it. They conjectured that a decimal multiplier whose whole part is clearly larger than its fractional part may be treated more like a whole number, as though the whole part "masks" or "absorbs" the fractional part. To support this conjecture, Fischbein et al. (1985) compared performance on several multiplication problems: one with the decimal multiplier 3.25, one with the decimal multiplier 1.25, and four with the decimal multipliers 0.75 or 0.65. They found that compared to decimals like 0.75, 0.65, or 1.25, a decimal like 3.25 has a slighter "counterintuitive" effect when playing the role of multiplier. This explanation raises several questions: What is the conceptual base for the argument that the whole part 3 in the decimal multiplier 3.25 better "masks" or "absorbs" the fractional part 0.25 than 1 does in the decimal multiplier 1.25? Is it merely a matter of the relative size between 3 and 1 or does a more fundamental factor account for the difference? If this is a matter of the relative size, would "large" decimals (e.g., 42.35) be better conceived as multipliers than small decimals (e.g., 3.25)? Would the "absorption effect" apply to decimal multipliers between 2 and 3 (e.g., 2.25)? Does the relative size between the whole part and the decimal part of a decimal multiplier play a role in the "absorption effect"? In this chapter we address some of these questions.

The third observation, related to the previous one, is that the notion of the "absorption effect" has not been investigated with respect to the intuitive partitive division rule, that the divisor must be a whole number, even though the same argument Fischbein et al. (1985) made with the multiplier can be made with the divisor. The argument would be that a non-whole-number divisor whose whole part is clearly greater than its fractional part should be treated like a whole number. That is, a partitive division problem with a divisor such as 2.53 is expected, according to the "absorption effect" notion, to be easier than a partitive division problem with a divisor such as 0.67. Therefore an additional question is, Does the "absorption effect" apply to partitive division as well?

TABLE 10.2. DISTRIBUTION OF RESPONSES TO DIVISION PROBLEMS

Problem	*Operation*	*% Correct (Grade)*
16	5 ÷ 15	20 (5), 24 (7), 41 (9)
17	5 ÷ 12	14 (5), 30 (7), 40 (9)
20	3.25 ÷ 5	73 (5), 71 (7), 84 (9)
21	.75 ÷ 5	85 (5), 77 (7), 83 (9)
22	1.25 ÷ 5	66 (5), 74 (7), 70 (9)

The fourth observation is that the intuitive rules do not seem to be equally robust in problem solutions. Consider, for example, Table 10.2, which shows the percentage distribution of responses to problems 16, 17, 20, 21, and 22 from Fischbein et al. (1985, p. 12). All these problems are of partitive division type and violate the same intuitive rule, that the divisor must be smaller than the dividend. Despite this uniformity, the results are strikingly different: The percentages of correct responses on Problems 16 and 17 are much lower than of those on Problems 20, 21, and 22. The explanation to this given by Fischbein et al. (1985) is that in Problems 16 and 17 the students' tendency was to reverse the roles of the numbers as a divisor and dividend. Had they done that in Problems 20 to 22, however, they would have ended up with a decimal divisor! It appears that, faced with having to cope with a violation of the partitive model's rules, the pupils chose instead not to reverse the numbers (p. 13). From this result we concluded that different intuitive rules within the partitive model may not be equally strong in affecting students' solution of partitive division problems: In Problems 20, 21, and 22 the children preferred to cope with the violation of the rule that the divisor must be smaller than dividend than with the violation of the rule that the divisor must be a whole number. Therefore the question of how different rule violations affect differently the choice of operation for solving multiplication and division problems needs to be extended to other intuitive rules.

In the rest of this chapter, we report on the instrument we developed that controls six confounding variables, used in this study with preservice and inservice elementary school teachers. Following this we discuss findings about the constraints imposed by the intuitive rules on the problem operators—multiplier and divisor—and the relative robustness of these constraints. We conclude with a summary and questions for further research.

METHOD

Subjects

The subjects who participated in this research were 167 inservice fourth–sixth grade teachers, 148 senior preservice teachers enrolled in a methods course for the teaching of elementary school mathematics (Group S), and 145 junior preservice teachers enrolled in a required sophomore-level content course in mathematics designed for preservice elementary school teachers (Group J). Both groups of students were declared majors in elementary education. The mathematical prerequisite for the methods course is the sophomore content course. For the content course the mathematical prerequisites are one year each of high school algebra and geometry.

Instrument

As has been indicated earlier, the instrument we developed for this study controls a wide range of confounding variables: number type, text, structure, context, syntax, and rule violation.

Number Type. This variable concerns the type of numbers representing the multipliers, multiplicands, divisors, and dividends; these were systematically varied across whole number, decimal greater than 1, and decimal less than 1.

Text. Two types of textual problems were included: mapping rule and multiplicative compare (à la Nesher, 1988). Mapping-rule problems are those that involve the phrase *for each*, such as, "For each child there are five bags of candies, there are four children, how many bags of candies are there altogether?"; multiplicative-compare problems are those that involve the phrase *times as many as*, such as, "Tom has five times as many candies as John, John has four candies, how many candies does Tom have?" For more on the conceptual differences between these two types of problems, see Harel, Post, and Behr (1988).

Structure. The third variable differentiates multiplicative problems according to an interpretation of their semantic structure: multiplication, partitive division, and quotitive division. Three types of mapping-rule problems were included: the *multiplication mapping-rule* type (e.g., "There are five shelves in Dan's room; Dan put eight books on each shelf; how many books are there in his room?") and the two division mapping-rule problems corresponding to it (e.g., "There are forty books in the room, and five shelves; how many books are there on each shelf if each shelf has the same number of books on it?" and "There are forty books in the room; eight books on each shelf; how many shelves are there?"). Three types of multiplicative-compare problems were also included: the multiplication multiplicative-compare problem (e.g., "Dan has twelve marbles; Ruth has six times as many marbles as Dan has; how many marbles does Ruth have?") and the two division multiplicative compare corresponding to it (e.g., "Ruth has seventy-two marbles; Ruth has six times as many marbles as Dan has; how many marbles does Dan have?" and "Ruth has seventy-two marbles; Dan has twelve marbles; how many times as many as Dan does Ruth have?").

Context. All problems used dealt with the familiar context of consumption; examples include the following problems: "Each pound of snow peas costs $3.00. Father buys 2.89 pounds of them. How many dollars does Father spend on snow peas?" and "Each child gets 24 ounces of milk. There are seven children. How many ounces of milk are needed?"

Syntax. This variable refers to the wording structure of the problem. More specifically, the syntactical structure of the problems used was controlled with regard to the number of the statements in the problem description, the location of the unknown quantity with respect to the other given quantities, and the coordination of units of measures (for more details see Harel and Behr, 1989; Harel et al., 1988; Conner, Harel, and Behr, 1988). All problems consisted of three statements: The first two were information statements and the third, a question statement. The coordination of the units of measures of the quantities involved was fixed across each one of problem types used.

Rule Violation. We classified multiplicative problems into eleven categories according to all possible rule violations described in Table 10.1. The instrument includes problems from each of these categories. With multiplication, there are three categories of problems:

- M(0) consists of problems that violate *none* of the intuitive rules in Table 10.1.
- M(1.1) consists of problems that violate only the rule that the multiplier must be a whole number. The multiplier in these problems must be a decimal greater than 1.
- M(1.2, 2) consists of problems that violate the two intuitive rules for multiplication: multiplier must be a whole number and multiplication makes bigger (or product must be larger than multiplicand); therefore the multiplier in these problems is necessarily smaller than 1.

With partitive division there are six categories:

- P(0) consists of problems that conform to all the intuitive partitive division rules in Table 10.1.
- P(1.1) consists of problems that violate *only* one rule: divisor must be a whole number. In these problems, therefore, the divisor is necessarily a non-whole number greater than 1.
- P(2) consists of problems that violate *only* the rule that the divisor must be smaller than the dividend.
- P(1.1, 2) consists of problems that violate *exactly* the two rules that the divisior must be a whole number and the divisor must be smaller than the dividend. The divisor in these problems is necessarily a non-whole number greater than 1 (to yet conform to quotient $<$ dividend) and greater than the dividend; that is, this category is the intersection of Categories P(1.1) and P(2).

- P(1.2, 3) consists of problems that violate *only* the two rules that the divisor must be a whole number and that division makes smaller. Consequently, the divisor in these problems is necessarily a number smaller than 1 and smaller than the dividend.
- P(1.2, 2, 3) consists of problems that violate the three partitive rules in Table 10.1. As a result, the divisor in these problems is a number smaller than 1 and bigger than the dividend; this category is the intersection of the Categories P(1.2, 3) and P(2).

With quotitive division there are two categories:

- Q(0) consists of problems that conform to the intuitive quotitive division rule in Table 10.1.
- Q(1) consists of problems that violate the only rule for quotitive division, that the divisor must be greater than the dividend.

A summary of this classification of problems according to rule violations is given in Table 10.3.

RESULTS AND DISCUSSION

Let us recall the main findings from studies investigating Fischbein et al.'s theory:

1. The relative difficulty of multiplication word problems is affected by the type of the multiplier, and the index for this relative difficulty is a multiplier represented by the number 1: Problems with a whole-number multiplier were easier than problems with a decimal multiplier larger than 1; and the latter were significantly easier than those with a (positive) decimal multiplier smaller than 1.
2. The multiplicand has no impact on problem difficulty.
3. The relative difficulty of partitive division word problems is affected by the type of the divisor and its order relation by magnitude to the dividend: Problems with a whole-number divisor smaller than the dividend yielded the highest level of performance; problems with a divisor greater than the dividend resulted in a decrease in the percentage of correct responses; and problems with a nonintegral decimal divisor

TABLE 10.3. CLASSIFICATION OF PROBLEM QUANTITIES ACCORDING TO RULE VIOLATIONS

Multiplicative situation	*Intuitive rule*	*Type of problem quantities violating the corresponding intuitive rule*	*Categories used*	*Example of of operation*
Multiplication	1. Multiplier must be a whole number	1.1. Multiplier is a non-whole number greater than 1	M(1.1)	16.5 × 3
	2. Multiplication makes bigger	1.2. Multiplier is a non-whole number smaller than 1	M(1.2, 2)	0.34 × 1.25
		Product is smaller than multiplicand		
Partitive division	1. Divisor must be a whole number greater than 1	1.1. Divisor is a non-whole number	P(1.1)	11 ÷ 2.53
		1.2. Divisor is a non-whole number smaller than 1	P(2)	3 ÷ 5
	2. Divisor must be smaller than dividend	Divisor is greater than dividend	P(1.1, 2)	12 ÷ 24.67
	3. Division makes smaller	Quotient is bigger than dividend	P(1.2, 3)	6 ÷ 0.67
			P(1.2, 2, 3)	0.35 ÷ 0.79
Quotitive division	1. Divisor must be smaller than dividend	1. Divisor is greater than dividend	Q(1)	83 ÷ 193

smaller than the dividend resulted in an even lower level of performance.

4. The relative difficulty of quotitive division word problems is affected by the order relation between the magnitudes of the divisor and the dividend: Problems with a divisor smaller than the dividend yielded a higher performance than those with a divisor greater than the dividend.

Some of our data are consistent with these findings and some do not fully agree with it; other data refine and even extend the observations about the previously documented cognitive obstacles to the solution of multiplicative problems. These are discussed in the following.

Multiplication. Table 10.4 shows the percentage distribution of responses to three categories of multiplication problems by the three samples of subjects who participated in this study: junior preservice teachers (Group J), senior preservice teachers (Group S), and inservice teachers. As can be seen, the percentage of correct responses on problems with a whole number multiplier (thus, conforming to Fischbein's multiplication model) is very high. On problems whose multiplier is a decimal greater than 1 (and therefore violating the rule that the multiplier must be a whole number) it drops slightly by an average of about 13 percent; and on problems whose multiplier is a decimal smaller than 1 (violating the exact same rule and the rule that the multiplication makes bigger) it drops by an average of about 41 percent. These results indicate a moderate effect on the level of difficulty from changing the type of multiplier from a whole number to a decimal greater than 1, but a great impact on the level of difficulty when the multiplier changes from a whole number or a decimal greater than 1 to a decimal smaller than 1. These results are consistent with results obtained in other studies, and therefore they support Fischbein et al.'s intuitive models for multiplication, the repeated addition model.

The percentage distribution of correct responses to the partitive division and quotitive division problems is included in Table 10.5. These results are consistent with the constraints of the models associated with these types of problems as were suggested by Fischbein et al. (1985). This can be seen by comparing the percentage of correct responses on P(0) prob-

TABLE 10.4. DISTRIBUTION OF RESPONSES TO MULTIPLICATION PROBLEMS

Type of Multiplier	*Type of Multiplicand*	*Example of operation*	*% Correct Responses (N)* — *Preservice* — *Group J*	*Preservice* — *Group S*	*Inservice*
Whole number	Whole number	14 × 8	96.5 (146)	94.5 (60)	98 (22)
	Decimal > 1	24 × 2.28	93 (150)	94.5 (60)	100 (22)
	Decimal < 1	15 × 0.38	95 (146)	97 (68)	99 (132)
		Mean	94.8	95.3	99
Decimal > 1	Whole number	16.5 × 3	82 (146)	80.5 (67)	91 (22)
	Decimal > 1	43.61 × 2.37	74 (145)	75.5 (72)	87 (132)
	Decimal < 1	2.5 × 0.75	79 (150)	89 (60)	96 (22)
		Mean	78.3	81.6	91.3
Decimal < 1	Whole number	0.45 × 150	69 (146)	78 (67)	77 (22)
	Decimal > 1	0.34 × 1.25	45 (150)	50 (60)	55 (22)
	Decimal < 1	0.75 × 0.6	45 (146)	41 (61)	40 (22)
		Mean	53	56.3	57.3

TABLE 10.5. DISTRIBUTION OF RESPONSES

Category	Rule(s) violated	Example of operation	Correct responses, % (N) Preservice Group J	Group S	Inservice
M(0)	No rule violation	24 × 2.28	93.6 (148)	95 (60)	99 (44)
M(1.1)	Multiplier must be a whole number	16.5 × 3	78.3 (137)	79.2 (60)	90.4 (55)
M(1.1, 2)	Multiplier must be a whole number *and* Multiplication makes bigger	0.75 × 0.62	51 (147)	55.8 (60)	58 (22)
P(0)	No rule violation	68 ÷ 17	86.5 (103)	91.5 (96)	94.5 (22)
P(1.1)	Divisor must be a whole number	11 ÷ 2.53	32.5 (62)	36 (94)	49 (77)
P(2)	Dividend must be greater than divisor	3 ÷ 5	64.3 (88)	69.3 (107)	78.5 (88)
P(1.1, 2)	Divisor must be a whole number *and* Dividend must be greater than divisor	12 ÷ 24.67	19.3 (88)	22 (103)	31 (44)
P(1.2, 3)	Divisor must be a whole number *and* Quotient must be greater than dividend	6 ÷ 0.67	44 (145)	46.3 (67)	52 (92)
P(1.1, 2, 3)	Divisor must be a whole number *and* Dividend must be greater than divisor *and* Quotient must be greater than dividend	0.35 ÷ 0.79	33 (146)	37 (67)	44 (131)
Q(0)	No rule violation	175 ÷ 35	90.6 (145)	91.5 (83)	95.1 (88)
Q(1)	Dividend must be greater than divisor	83 ÷ 193	38.4 (145)	35.2 (89)	63.7 (39)

	Dividend must be greater than divisor *and* Quotient must be greater than dividend	0.35 ÷ 0.79	33 (146)	37 (67)	44 (131)
Q(0)	No rule violation	175 ÷ 35	90.6 (145)	91.5 (83)	95.1 (88)
Q(1)	Dividend must be greater than divisor	83 ÷ 193	38.4 (145)	35.2 (89)	63.7 (39)

lems (those that conform to the partitive division model) to P(1.1), P(2), P(1.1, 2), P(1.2, 3), and P(1.2, 2, 3) problems (those that violate some or all of the partitive division rules), and the percentage of correct responses on Q(0) problems (those that conform to the quotitive division model) to Q(1) (those that violate the rule that the dividend must be greater than the divisor): The performance on P(0) problems is significantly higher than the performance on each of the other partitive division categories, except P(2); and the performance on Q(0) problems is significantly higher than the performance on Q(1) problems.

These findings are consistent with other studies' findings; therefore they support Fischbein et al.'s argument that intuitive models, which are not necessarily in accord with the operations of multiplication and division, govern subjects' solutions of multiplicative problems. However, as is discussed later, a further look at our data leads to new observations.

The Impact of the Operators, Multiplier and Divisor. Recall Fischbein et al.'s "absorption effect" notion that they suggested to explain the conceptual basis for why multiplicative problems with "small" (e.g., 0.65 and 1.25) decimal operators (i.e., multipliers and divisors) are more difficult than those with "big" (e.g., 3.25) decimal operators: "We conjectured that when the whole part of a decimal is clearly larger than the fractional part, the pupil may treat it more like a whole number (as though the whole part 'masks' or 'absorbs' the fractional part). . . . Although the decimal operator still appears as a source of difficulty, one can see that compared to decimals like 0.75, 0.65, or 1.25, a decimal like 3.25 has a slighter counterintuitive effect when playing the role of operator" (p. 11). From this explanation it is expected that an increase in the value of a decimal operator (multiplier or divisor) greater than 1 will increase the effect of masking the decimal part of the operator by its whole number part, which, in turn, will decrease problem difficulty. As can be seen in Table 10.6, our data on subjects' performance on multiplication problems do *not* dovetail with this pattern: An increase in the whole part of the multiplier has not led to a decrease in problem difficulty. Further, the data in Table 10.5 suggest

TABLE 10.6. DISTRIBUTION OF RESPONSES TO MULTIPLICATION PROBLEMS WITH DECIMAL MULTIPLIERS GREATER THAN 1

		% Correct responses (N)		
		Preservice		
Multiplier	*Multiplicand*	*Group J*	*Group S*	*Inservice*
1.45	2.86	71(93)	—	87(23)
2.5	0.75	87(150)	90(60)	96(22)
2.89	3	88(146)	89(62)	91(22)
10.5	18.25	94(145)	93(72)	100(22)
16.5	3	76(146)	79(67)	91(22)
43.61	2.37	54(146)	58(72)	74(132)

that the "absorption effect" does not apply to partitive division problems. This can be seen by comparing the performances of problems whose divisor is a decimal greater than 1 (P(1.1)) and problems whose divisor is a decimal smaller than 1 (P(1.2, 3)). In both cases the level of performance is relatively low compared to the level of performance on P(0) problems (those which conform to Fischbein et al.'s partitive model).

Further, note that the performance on the division problem with the divisor 24.67 ((P1.1,2)) is very low (less than 30 percent; see Table 10.5), even though its whole part is relatively large. One might argue that this low result is attributable to the fact that this problem violates another rule, that the divisor must be smaller than dividend. Although it is likely that this rule violation has, to some extent, affected subjects' performance, we do not think that its effect is that strong. This is supported by the fact that the performance on problems that violate only this rule (i.e., P(2)) is relatively high (Table 10.5).

Levels of Robustness. In Table 10.5, there are two conspicuous results concerning how the intuitive rules effect differently the solution performance: First, the performance on M(1.1) problems (Table 10.5) that violate only the rule that the multiplier must be a whole number is higher than the performance on M(1.2, 2) problems that violate two rules:

that rule and the rule that multiplication makes bigger. Second, the performance on P(2) problems (those that violate only the rule that the dividend must be greater than the divisors) is higher than the performance on the other categories of partitive division problems that violate one or a combination of intuitive rules. The latter result indicates that, when the divisor is a whole number, the rule that the dividend must be greater than the divisor is the least robust in the solution of partitive division among other combinations of the intuitive rules. This is not surprising, because problems with this rule violation (i.e., a number of objects, x, is to be equally divided into a whole number of sets, y, where $x < y$) are quite common in everyday situations. On the other hand, problems that violate the rule that the divisor must be greater than the dividend and, in addition, violate the rule that the divisor must be a whole number (i.e., P(1.1, 2)) scored the lowest level of performance.

In looking at the subjects' solutions to problems in Category P(1.1), we found that almost all subjects who did not solve these types of problems correctly offered an inverse expression to the correct division expression (for example, if the solution of the problem was $11 \div 2.53$, the incorrect solution was $2.53 \div 11$). Many more subjects chose the inverse expression (mean = 54.75) than solved the problem correctly (mean = 32.5). Note that, although the correct expression violates the rule that the divisor must be a whole number, the inverse expression violates the rule that the divisor must be smaller than the dividend. We interpret this result to indicate that the former rule is more robust that the latter one.

The Impact of the Multiplicand. Looking back at Table 10.4, one can see that our data do not fully dovetail with other studies that have observed no impact of the multiplicand. Indeed, consistent with other studies, Table 10.4 shows that, as long as the multiplier is a whole number or a decimal greater than 1, the type of the multiplicand as a whole number, decimal greater than 1, or decimal smaller than 1, has almost no effect on the problem's difficulty. On the other hand, when the multiplier is smaller than 1, our data show that the type of the multiplicand seems to have some effect: The percentage of correct responses on problems with a whole-number multiplicand is much higher than that on problems with decimal multiplicand.

This observation could, of course, be a result of a perturbation or noise in our data. However, assuming this is not so, we speculate the following explanation: The presence of a whole-number quantity in the problem helps subjects to sort out the role of the quantities involved, which, in turn, enables them to choose a correct operation for solving the problem.

SUMMARY

In this chapter, we dealt with several aspects of the impact of the number type on the relative difficulty of multiplicative problems. We reexamined the findings from other studies concerning this impact, investigated the "absorption effect" notion suggested by Fischbein et al. to account for differences in subjects' performance on multiplicative problems with different non-whole-number operators and probed into the level of robustness of the intuitive rules derived from Fischbein et al.'s models. The observations and findings reported in this chapter are summarized as follows:

1. The instruments used in Fischbein et al. and the studies that followed it do not control for many variables known to be influential in problem solution. We offered a framework for an instrument that controls for a wide range of confounding variables: number type, text, structure, context, syntax, and rule violation.
2. Our data is consistent with the finding that subjects' model for multiplication is the repeated addition model, and for division, subjects' models are partitive division and quotitive division.
3. Our data do *not* support Fischbein et al.'s notion of the "absorption effect": No significant difference in performance was found between multiplication problems with multipliers whose whole part is relatively large and those with multipliers whose whole part is relatively small. Moreover, the absorption effect does not apply to division problems.
4. Evidence was shown for differential robustness of the intuitive rules associated with Fischbein et al.'s models.
5. The type of multiplicand seems to have an impact on problem solution when the multiplier is smaller than 1.

FURTHER RESEARCH QUESTIONS

This research has raised several questions for further investigations. First, this and other studies focused on one type of problem quantities: decimal numbers. The question of whether subjects encounter similar difficulties with multiplication and division problems that involve fractions has never been directly addressed. There is a reason to believe, however, that fractions and decimals do not have the same effect on the solution of multiplication and division problems. More specifically, it seems easier to solve multiplication and division problems in which the operator (i.e., the multiplier or divisor) is a fraction than in those in which the operator is a decimal. A rationale for this is based on the fact the naming rule of fractions is different from the naming rule of decimals: Under these naming rules, it is easier to identify the role of a problem quantity as an operator or operand if the quantity is a fraction than if it is a decimal; therefore it is easier to recognize its relation to other problem quantities. For example, the two propositions in the statement, "*John had 5 ounces of ice cream* and *he ate* x *of the amount he had*" are easier to connect if (the operator) x is a fraction, say 2/5, than if x is a decimal, say 0.40.

Second, a further distinction among the intuitive rules derived from Fischbein et al.'s model is that some of the rules are associated with the *problem information*, others with the *problem solution*. In multiplication, the rule that the multiplier must be a whole number imposes a constraint on the type of multiplier provided in the problem information; in contrast, the rule that the multiplication makes bigger restricts the problem solution to be a number greater than the multiplicand. Similarly, in partitive division, the rules that the divisor must be a whole number and the divisor must be smaller than dividend are problem information rules, whereas the rule that the quotient must be greater than dividend is a problem solution rule. Finally, in quotitive division, the rule that the divisor must be smaller than dividend is a problem information rule; no problem solution rule is involved. This raises the question of whether problem information rules are equally robust as the problem solution rules.

Finally, when we looked at the other studies' data on multiplication and division, we raised the question, Why are problems with a multiplier greater than 1 relatively easy for the subjects despite the fact that they are in conflict with the model of multiplication as a repeated addition? If indeed this model governs subjects solution of multiplication problems, it is not at all clear why the intuitive rule derived from it—that the multiplier must be a whole number—is substantially less robust in the case of a non-whole-number multiplier greater than 1 than in the case of a multiplier smaller than 1. Further, it is not all clear what is the conceptual basis for the multiplier 1 being an index for the relative difficulty of multiplication problems. In fact, Fischbein et al. in their explanation to the observation that the intuitive rule that the multiplier must be a whole number does not equally affect multiplication problems with decimal multipliers, did not differentiate between multipliers greater than 1 and those less than 1. Rather, they suggested the "absorption effect" notion, which differentiates between multiplicative problems according to the size relationship between the whole number part and the fractional part of their multipliers. In this chapter we reported data that are *not* consistent with this explanation; therefore, further theoretical and empirical investigations are needed to answer these questions.

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