MATH 140B - HW 0 SOLUTIONS

Problem 1 (WR Ch 5 #1)**.** Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is constant.

Solution. For $x \neq y$, from the above inequality we have $\frac{|f(x)-f(y)|}{|x-y|} \leq |x-y|$. So then

$$|f'(y)| = \left|\lim_{x\to y} \frac{f(x)-f(y)}{x-y}\right| = \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| \leq \lim_{x\to y}|x-y| = 0.$$

This implies that $f'(y) = 0$ for all $y \in \mathbb{R}$, so f is constant.

Problem 2 (WR Ch 5 #3)**.** Suppose g is a real function on $\mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough.

Solution.

$$f \text{ is one-to-one} \qquad \Longleftrightarrow \qquad \forall a, b \in \mathbb{R}, \quad a \neq b \ \Rightarrow \ f(a) \neq f(b).$$

Assume without loss of generality that $a < b$. Then by the Mean Value Theorem, there exists some $c \in (a, b)$ such that $g(b) - g(a) = (b-a)\ g'(c)$. Then we have

$$\begin{aligned} f(b) - f(a) &= (b - \epsilon g(b)) - (a - \epsilon g(a)) \\ &= (b-a) - \epsilon(g(b) - g(a)) \\ &= (b-a) - \epsilon(b-a)g'(c) \\ &= (b-a)(1-\epsilon g'(c)). \qquad (*) \end{aligned}$$

In this last expression $(b-a) \neq 0$ since $a < b$, and if we let $\epsilon < \frac{1}{M}$, then

$$|\epsilon g'(c)| < \frac{1}{M}|g'(c)| \leq \frac{1}{M}\ M = 1,$$

so $(1-\epsilon g'(c)) \neq 0$. This proves that $(*)$ is nonzero, so $f(b) - f(a) \neq 0$, and thus $f(a) \neq f(b)$, completing the proof.

Problem 3 (WR Ch 5 #5)**.** Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution. By the Mean Value Theorem, there exists some $y_x \in (x, x+1)$ (we write y_x because to indicate that y_x depends on x) such that

$$f(x+1) - f(x) = ((x+1) - x) f'(y_x) = f'(y_x).$$

Since the left hand side is $g(x)$, we have

$$\lim_{x\to+\infty} g(x) = \lim_{x\to\infty} f'(y_x) = \lim_{y\to\infty} f'(y) = 0.$$