MATH 140B - HW 1 SOLUTIONS

Problem 1 (WR Ch 5 #6). Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing,

Put
$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$
and prove that g is monotonically increasing.

Solution. If we can prove that $g'(x) > 0$ for $x > 0$, then this will show that g is monotonically increasing (by Theorem 5.11a). By the quotient rule,
$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \qquad (x > 0),$$
so all we need to show is that $xf'(x) - f(x) > 0$ for $x > 0$. By properties (a) and (b) we can use the Mean Value Theorem, which says there exists some $y \in (0, x)$ such that
$$f(x) - f(0) = (x - 0)f'(y_x) = xf'(y),$$
and by property (d), since $y < x$ then $f'(y) < f'(x)$. Also using property (c), we have
$$f(x) = f(x) - f(0) = xf'(y) < xf'(x) \qquad \Longrightarrow \qquad xf'(x) - f(x) > 0,$$
finishing the proof.

Problem 2 (WR Ch 5 #7). Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that
$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Solution.
$$\frac{f'(x)}{g'(x)} = \frac{\lim_{t\to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t\to x} \frac{g(t)-g(x)}{t-x}} = \lim_{t\to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-g(x)} = \lim_{t\to x} \frac{f(t)-0}{g(t)-0} = \lim_{t\to x} \frac{f(t)}{g(t)}$$

Problem 3 (WR Ch 5 #8)**.** Suppose f' is continuous on $[a,b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \epsilon$$

whenever $0 < |t-x| < \delta$, $a \le x \le b$, $a \le t \le b$. Does this hold for vector-valued functions too?

Solution. Remember that a continuous function on a compact set is uniformly continuous. Since f' is continuous on $[a,b]$, it is uniformly continuous, which means for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |y-x| < \delta \qquad \Longrightarrow \qquad |f'(y) - f'(x)| < \epsilon. \tag{$*$}$$

Given any $t, x \in [a,b]$ such that $0 < |t-x| < \delta$ and $t > x$, by the Mean Value Theorem we can find some $y \in (x,t)$ such that $f(t) - f(x) = (t-x)f'(y)$, which means

$$f'(y) = \frac{f(t)-f(x)}{t-x}.$$

Notice also that $y \in (x,t)$ and $0 < |t-x| < \delta$ imply that $0 < |y-x| < \delta$, so by $(*)$ we have

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \epsilon. \tag{$**$}$$

Now we ask if this holds in general for vector-valued functions. Let $f(x) = \langle f_1(x), \ldots, f_n(x)\rangle$. If f' is continuous on $[a,b]$, then f_i is a continuous real-valued function on $[a,b]$ for each $1 \le k \le n$, so for each one we can choose a $\delta_i > 0$ such that if $0 < |x-t| < \delta_i$ then

$$\left|\frac{f_i(t)-f_i(x)}{t-x} - f_i'(x)\right| < \frac{\epsilon}{\sqrt{n}}.$$

Then if we let $\delta = \min\{\delta_1, \ldots, \delta_n\}$, if $0 < |t-x| < \delta$, then

$$\begin{aligned}
\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| &= \left|\frac{\langle f_1(t),\ldots,f_n(t)\rangle - \langle f_1(x),\ldots,f_n(x)\rangle}{t-x} - \langle f_1'(x),\ldots,f_n'(x)\rangle\right| \\
&= \left|\left\langle \frac{f_1(t)-f_1(x)}{t-x} - f_1'(x), \ldots, \frac{f_n(t)-f_n(x)}{t-x} - f_n'(x)\right\rangle\right| \\
&= \sqrt{\left(\frac{f_1(t)-f_1(x)}{t-x} - f_1'(x)\right)^2 + \ldots + \left(\frac{f_n(t)-f_n(x)}{t-x} - f_n'(x)\right)^2} \\
&< \sqrt{\frac{\epsilon^2}{n} + \ldots + \frac{\epsilon^2}{n}} \quad = \quad \epsilon.
\end{aligned}$$

Problem 4 (WR Ch 5 #9)**.** Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Solution. It does indeed follow. Since f is continuous, $\lim_{x \to 0} f(x) = f(0)$, so

$$\lim_{x \to 0} [f(x) - f(0)] = 0.$$

Similarly, $\lim_{x \to 0} x = 0$. This allows us to use L'Hôpital's rule to say that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f'(x)}{1} = \lim_{x \to 0} f'(x) = 3,$$

so $f'(0)$ exists.

Problem 5 (WR Ch 5 #22a)**.** Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$. If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Solution. Assume by way of contradiction that f has two fixed points a and b. Without loss of generality we can assume $a < b$ (by calling the smallest one x). By the Mean Value Theorem, there exists some $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)\ f'(c).$$

But since a and b are fixed points we have $f(a) = a$ and $f(b) = b$, so our equation becomes

$$(b - a) = (b - a)\ f'(c),$$

and since $a < b$, we can divide by $(b - a)$ to get $f'(c) = 1$, a contradiction.