[3]

Large Values of x and the Multiplication Formula

Can we find an elementary function that gives an accurate approximation of $\Gamma(x)$ for large values of x? If the growth of $n!$ is estimated, it is found to increase with n faster than $n^n e^{-n}$, but not quite as fast as $n^{n+1}e^{-n}$.* In other words, the growth of $\Gamma(n)$ is caught between $n^{n-1}e^{-n}$ and $n^n e^{-n}$. This suggests that we consider a function of the form

$$f(x) = x^{x-1/2}\, e^{-x}\, e^{\mu(x)}, \tag{3.1}$$

in order to study the behavior of $\Gamma(x)$ for large x. Our goal is to make $f(x)$ satisfy the basic conditions for the gamma function by choosing $\mu(x)$ in an appropriate way.

If we replace x by $x + 1$ in Eq. (3.1) and divide the resulting expression by Eq. (3.1), we get

$$\frac{f(x+1)}{f(x)} = \left(1 + \frac{1}{x}\right)^{x+1/2} x e^{-1}\, e^{\mu(x+1)-\mu(x)}.$$

This shows that $f(x)$ satisfies condition (1) in Theorem 2.1 if, and only if,

$$\mu(x) - \mu(x+1) = (x + \tfrac{1}{2}) \log\left(1 + \frac{1}{x}\right) - 1, \tag{3.2}$$

holds for $\mu(x)$.

* If we consider the elementary inequalities

$$\left(1 + \frac{1}{k}\right)^k < e < \left(1 + \frac{1}{k}\right)^{k+1}$$

for $k = 1, 2, \cdots, (n-1)$, and multiply them together, we get

$$\frac{n^{n-1}}{(n-1)!} < e^{n-1} < \frac{n^n}{(n-1)!}.$$

This leads to the approximation

$$en^n e^{-n} < n! < en^{n+1}e^{-n}.$$

We denote the right side of Eq. (3.2) by $g(x)$. A function $\mu(x)$ with this property is easy to find. If we set

$$\mu(x) = \sum_{n=0}^{\infty} g(x+n), \tag{3.3}$$

then Eq. (3.2) holds, provided the infinite series in Eq. (3.3) converges. Let us postpone the proof of convergence for a moment and consider condition (2) of theorem 2.1.

The factor $x^{x-1/2} e^{-x}$ in Eq. (3.1) is log convex because the second derivative of its logarithm, $1/x + \frac{1}{2}x^2$, is always positive when x is positive. If we can show that the factor $e^{\mu(x)}$ is log convex, in other words that $\mu(x)$ is convex, then $f(x)$ also satisfies condition (2). This means that the function $f(x)$ determined by the particular $\mu(x)$ we defined in Eq. (3.3) will agree with $\Gamma(x)$ to within a constant factor. Our $\mu(x)$ is convex if the general term of the series $g(x+n)$ is convex. To show this, it suffices to prove the convexity of $g(x)$ itself. But we have

$$g''(x) = \frac{1}{2x^2(x+1)^2} > 0.$$

The convergence of the series in Eq. (3.3) still remains to be shown. We will combine this with an approximation of the function $\mu(x)$. Let us begin by considering the expansion

$$\tfrac{1}{2}\log\frac{1+y}{1-y} = \frac{y}{1} + \frac{y^3}{3} + \frac{y^5}{5} + \cdots,$$

which is valid for $|y| < 1$. Now we replace y by $1/(2x+1)$. The resulting expansion is valid for positive x because $1/(2x+1) < 1$ whenever $x > 0$. We multiply this equation by $2x + 1$ and bring the first term on the right side over to the left side:

$$(x + \tfrac{1}{2})\log\left(1 + \frac{1}{x}\right) - 1 = g(x)$$

$$= \frac{1}{3(2x+1)^2} + \frac{1}{5(2x+1)^4} + \frac{1}{7(2x+1)^6} + \cdots.$$

This expression again shows that $g(x)$ is convex, since every term on the right side is convex. Now we can approximate $g(x)$. If the integers $5, 7, 9, \cdots$ are all replaced by 3, then the value of the right side increases. The result is an infinite geometric series, having $1/(3(2x+1)^2)$ as its first term and $1/(2x+1)^2$ as its ratio. Its sum is

$$\frac{1}{3(2x+1)^2}\,\frac{1}{1-(1/(2x+1)^2)} = \frac{1}{12x(x+1)} = \frac{1}{12x} - \frac{1}{12(x+1)}.$$

But $g(x)$ is positive, hence

$$0 < g(x) < \frac{1}{12x} - \frac{1}{12(x+1)}.$$

Since every term of the series in Eq. (3.3) is positive, it suffices to show the convergence of

$$\sum_{n=0}^{\infty} \left(\frac{1}{12(x+n)} - \frac{1}{12(x+n+1)} \right),$$

which converges trivially to the limit $1/12x$. This not only proves our assertion, it also gives the approximation

$$0 < \mu(x) < \frac{1}{12x}.$$

In other words,

$$\mu(x) = \frac{\theta}{12x}$$

where θ is a number independent of x between 0 and 1.

By a suitable choice of the constant a, we get

$$\Gamma(x) = ax^{x-1/2}\, e^{-x+\mu(x)} = ax^{x-1/2}\, e^{-x+\theta/12x}. \tag{3.4}$$

If we let x be an integer n and multiply the expression by n, we get the approximation

$$n! = an^{n+1/2}\, e^{-n+\theta/12n}. \tag{3.5}$$

We are now going to find the exact value of this constant a and determine some other important constants at the same time.

Let p be a positive integer. We consider the function

$$f(x) = p^x \Gamma\left(\frac{x}{p}\right) \Gamma\left(\frac{x+1}{p}\right) \cdots \Gamma\left(\frac{x+p-1}{p}\right),$$

for $x > 0$. The second derivative of $\log p^x$ is zero, and each of the functions $\Gamma((x+i)/p)$ is obviously log convex. This implies that $f(x)$ is also log convex. If we replace x by $x + 1$, p^x takes on the factor p, $\Gamma((x+i)/p)$ goes over into the next factor, and $\Gamma((x+p-1)/p)$ becomes

$$\Gamma\left(\frac{x}{p} + 1\right) = \frac{x}{p}\, \Gamma\left(\frac{x}{p}\right).$$

In other words, $f(x)$ is multiplied by x. Our function again satisfies the conditions (1) and (2) in Theorem 2.1; therefore,

$$p^x \Gamma\left(\frac{x}{p}\right) \Gamma\left(\frac{x+1}{p}\right) \cdots \Gamma\left(\frac{x+p-1}{p}\right) = a_p \Gamma(x), \tag{3.6}$$

where a_p is a constant depending on p. For $x = 1$ in Eq. (3.6), we have

$$a_p = p\Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{2}{p}\right) \cdots \Gamma\left(\frac{p}{p}\right). \tag{3.7}$$

If we set $x = k/p$ in Eq. (2.7), then a simple manipulation gives

$$\Gamma\left(\frac{k}{p}\right) = \lim_{n\to\infty} \frac{n^{k/p} n!\, p^{n+1}}{k(k+p)(k+2p)\cdots(k+np)}.$$

Now we set $k = 1, 2, \cdots, p$, one after the other, and multiply all these expressions together. Factors of the form $(k + hp)$ appear in the denominator, where k runs from 1 to p, and h runs from 0 to n. For $h = 0$ we get the numbers from 1 to p; for $h = 1$, the numbers from $p + 1$ to $2p$; and so on. The product in the denominator is obviously $(np + p)!$. The final result is

$$a_p = p \lim_{n\to\infty} \frac{n^{(p+1)/2} (n!)^p\, p^{np+1}}{(np+p)!}.$$

The well-known infinite product

$$1 = \lim_{n\to\infty} \left(1 + \frac{1}{np}\right)\left(1 + \frac{2}{np}\right) \cdots \left(1 + \frac{p}{np}\right),$$

which can be written as

$$1 = \lim_{n\to\infty} \frac{(np+p)!}{(np)!\,(np)^p},$$

can now be applied. If we multiply this last expression with the above identity for a_p, we obtain

$$a_p = p \lim_{n\to\infty} \frac{(n!)^p\, p^{np}}{(np)!\, n^{(p-1)/2}}.$$

But Eq. (3.5) implies that

$$(n!)^p = a^p n^{np+p/2}\, e^{-np}\, e^{\theta_1 p/12n},$$

$$(np)! = a(np)^{np+1/2}\, e^{-np}\, e^{\theta_2/12np}.$$

After making the appropriate substitutions above, we obtain

$$a_p = \sqrt{p}\, a^{p-1} \lim_{n\to\infty} e^{(\theta_1 p/12n)-(\theta_2/12np)},$$

and finally

$$a_p = \sqrt{p}\, a^{p-1}. \tag{3.8}$$

By evaluating a_2 with the help of Eq. (3.7) and then comparing the result with Eq. (3.8), we get

$$a_2 = 2\Gamma(\tfrac{1}{2})\,\Gamma(1) = 2\sqrt{\pi} = a\sqrt{2}.$$

But this determines the exact values of our constants:

$$a = \sqrt{2\pi} \quad \text{and} \quad a_p = p^{1/2}(2\pi)^{(p-1)/2}.$$

Now we gather together all the important expressions from this chapter:

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-1/2}\, e^{-x+\mu(x)},$$

$$\mu(x) = \sum_{n=0}^{\infty} (x + n + \tfrac{1}{2}) \log\left(1 + \frac{1}{x+n}\right) - 1 = \frac{\theta}{12x}, \qquad 0 < \theta < 1,$$

$$n! = \sqrt{2\pi}\, n^{n+1/2}\, e^{-n+\theta/12n}. \tag{3.9}$$

$$\Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\cdots\Gamma\left(\frac{x+p-1}{p}\right) = \frac{(2\pi)^{(p-1)/2}}{p^{x-1/2}}\,\Gamma(x). \tag{3.10}$$

In particular, for $p = 2$

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\,\Gamma(x). \tag{3.11}$$

The formulas in Eq. (3.9), which describe the behavior of $\Gamma(x)$ for large values of x, are called *Stirling's formulas*. If our approximation of $\mu(x)$ is used, the accuracy of the formula for $\Gamma(x)$ will increase as x increases. This is also true for estimates of $n!$ The relative accuracy for $n \geqslant 10$ is already quite high.

The functional equation (3.10), discovered by Gauss, is called *Gauss' multiplication formula*. By replacing x by px in Eq. (3.10), we obtain an expression for $\Gamma(px)$ as the product of factors, each of the form $\Gamma(x + (k/p))$. This fact gave rise to the name "multiplication formula." The most important special case is $p = 2$. It was discovered by Legendre and is often referred to as *Legendre's relation*.

[4]

The Connection with sin x

The gamma function satisfies another very important functional equation. In order to derive it, we set

$$\varphi(x) = \Gamma(x)\,\Gamma(1-x)\sin \pi x. \tag{4.1}$$

This function is only defined for nonintegral arguments. If we replace x by $x + 1$, then $\Gamma(x)$ becomes $x\Gamma(x)$. The function $\Gamma(1-x)$ becomes

$$\Gamma(-x) = \frac{\Gamma(1-x)}{-x},$$

and $\sin \pi x$ changes its sign. This means that $\varphi(x)$ is left fixed, and is therefore periodic of period 1:

$$\varphi(x+1) = \varphi(x). \tag{4.2}$$

The Legendre relation can be written in the form

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = b2^{-x}\,\Gamma(x),$$

where b is a constant. Actually, the exact value of b was determined in Chapter 3. But this extra information need not (and will not) be assumed here. As far as we are concerned now, b is just some particular constant.

In the expression above, we replace x by $1 - x$:

$$\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(1 - \frac{x}{2}\right) = b2^{x-1}\,\Gamma(1-x).$$

Now we consider

$$\varphi\left(\frac{x}{2}\right)\varphi\left(\frac{x+1}{2}\right) = \Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)\sin\frac{\pi x}{2}\,\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\cos\frac{\pi x}{2}$$
$$= \frac{b^2}{4}\,\Gamma(x)\,\Gamma(1-x)\sin \pi x,$$

and we get the relation

$$\varphi\left(\frac{x}{2}\right)\varphi\left(\frac{x+1}{2}\right) = d\varphi(x), \tag{4.3}$$