On the Number e

E. Kuz'min and A. I. Shirshov

1. What Is e?

You have probably already been introduced to the number π (pi), a number with a very long history, whose beginnings are lost in the ancient mists of time.[1] The number π appears in high-school courses, but in high school you may never encounter the number e at all. However, this wonderful number, which came into use in the eighteenth century with the development of mathematical analysis, plays a role in modern mathematics that is perhaps even more important than that of π.

A typical textbook gives two definitions of the number e. The second (we won't be needing the first) defines e as the limit of the sequence $x_n = \left(1+\frac{1}{n}\right)^n$. Let us prove that this limit exists.

Lemma 1. *For all m, n*

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^{m+1}. \tag{1}$$

The inequality (1) is very easily proved using the well-known *Cauchy inequality*

$$\sqrt[n]{a_1 a_2 \cdots a_n} < \frac{a_1+a_2+\cdots+a_n}{n}, \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are positive numbers, not all the same.[2] From (2) it follows that

$$\sqrt[m+n+1]{\left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{m+1}\right)^{m+1}} < \frac{n\left(1+\frac{1}{n}\right)+(m+1)\left(1-\frac{1}{m+1}\right)}{m+n+1} = 1,$$

from which we obtain

$$\left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{m+1}\right)^{m+1} < 1$$

and

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^{m+1}.$$

The Russian original is published in *Kvant* 1979, no. 8, pp. 3–8.

[1] Many interesting facts about the number π can be found in F. Kympan's book *A History of the Number π* (Moscow, Nauka, 1971). See also the article by A. Zvonkin, "What Is π?" (*Kvant* 1978, no. 11).

[2] Try proving this on your own.

LEMMA 2. *The sequence $\{x_n\}$ is an increasing sequence.*

PROOF. Let us denote $\left(1+\frac{1}{n}\right)^{n+1}$ by y_n. According to Lemma 1, $x_{n(n+2)} < y_{n+1}$; that is,

$$\left(1+\frac{1}{n(n+2)}\right)^{n(n+2)} < \left(1+\frac{1}{n+1}\right)^{n+2}.$$

Therefore,

$$\left(1+\frac{1}{n(n+2)}\right)^{n} < \left(1+\frac{1}{n+1}\right).$$

Consequently,

$$\left(1+\frac{1}{n(n+2)}\right)^{n} = \frac{(n+1)^{2n}}{n^n(n+2)^n} < \frac{n+2}{n+1}.$$

As a result,

$$\left(\frac{n+1}{n}\right)^{n} < \left(\frac{n+2}{n+1}\right)^{n+1},$$

that is,

$$x_n < x_{n+1}. \qquad \square$$

From Lemmas 1 and 2 and from Weierstrass's theorem it follows that the sequence $\{x_n\}$ approaches a limit. The notation e to denote this limit was introduced by Leonhard Euler (1707–1783).

EXERCISE. Prove that $\lim_{n\to\infty} y_n = e$ and that for all m, n,

$$x_n < e < y_m.$$

The number e can also be defined as the limit of the sequence

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Let us prove this. According to Newton's formula,

$$\begin{aligned} x_n &= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \cdots \\ &\quad + \frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{1}{n^k} + \cdots + \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots \\ &\quad + \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots \\ &\quad + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \end{aligned} \tag{3}$$

(From (3) it can also be seen that the sequence $\{x_n\}$ is an increasing sequence.) Eliminating the factors in parentheses simply increases each item on the right-hand side of (3). Therefore, $x_n < s_n$ (when $n > 1$).

We now fix an arbitrary $k > 1$. Assuming that $n > k$, we discard all the items on the right-hand side of equation (3), beginning with the $(k+2)$th. We thus obtain

$$x_n > 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right). \tag{4}$$

As $n \to \infty$ (for fixed k), the right-hand side of inequality (4) approaches s_k. Passing to the limit in (4), we obtain $e \geq s_k$. Thus, $x_n < s_n \leq e$. From $\lim_{n\to\infty} x_n = e$ we obtain $\lim_{n\to\infty} s_n = e$. In this way,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots. \tag{5}$$

It is much more convenient to use the sequence $\{s_n\}$ for computing approximate values of the number e than it is to use the sequence $\{x_n\}$. Let us determine the difference $e - s_n$. From (5) it follows that

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \\ &= \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right] \\ &< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots\right] \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+2}} = \frac{n+2}{(n+1)^2 \cdot n!} \\ &< \frac{1}{n \cdot n!}. \end{aligned}$$

The result obtained can be expressed in the form of the following equation:

$$e = s_n + \frac{\theta_n}{n \cdot n!} \quad (0 < \theta_n < 1). \tag{6}$$

From (6) it is easy to show that the number e is irrational. Let us see what happens if we assume that e is rational, that is, that $e = \frac{p}{q}$ for some integers p and q. Then $q! \cdot e$ is an integer. It follows from (6), with $n = q$, that

$$q! \cdot e = q! + q! + \frac{q!}{2!} + \cdots + \frac{q!}{q!} + \frac{\theta_q}{q}.$$

It follows that $\frac{\theta_q}{q}$ is also an integer, which is obviously not the case, and so our assumption that e is rational was incorrect. Here are the first decimal digits of the irrational number e:

$$e = 2.718281828459045\ldots.$$

2. The Problem of Partitions.

The number e, the darling of mathematical analysis and the theory of functions, pops up rather unexpectedly in certain combinatorial problems. Let us look at one such problem.

We begin with an example. A set with two elements $\{a_1, a_2\}$ can be divided into disjoint nonempty subsets (in talking about partitions, such subsets are called *classes*) in two ways: There's the "element-by-element" partition $\{a_1\}, \{a_2\}$ and the degenerate "whole" partition $\{a_1, a_2\}$. Analogous partitions can be made with any set of n elements ($n > 2$). When $n = 3$, three more kinds of partitions become possible (Figure 1).

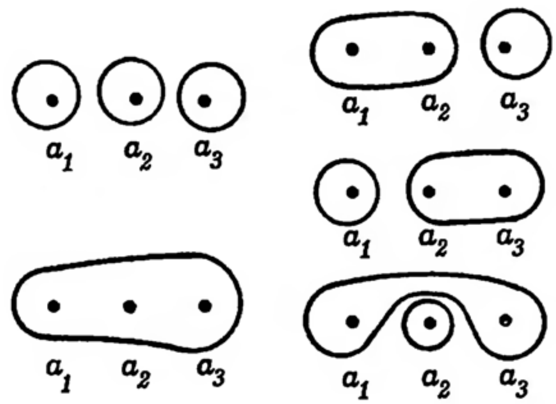

FIGURE 1

Let us call the number of partitions of an n-element set $\tau(n)$.[3] Clearly, $\tau(0) = \tau(1) = 1$. As we have just seen, $\tau(2) = 2, \tau(3) = 5$. And what is the value of $\tau(n)$ for an arbitrary n? Is it possible to find a simple formula for $\tau(n)$?

As often happens in combinatorial problems, it is easy to find a recurrence relation connecting $\tau(n)$ with the values $\tau(k)$ for $k < n$.

Let $M = \{a_1, a_2, \ldots, a_n\}$. Let us arrange all the partitions of the set M according to the number k of elements in the class into which the element a_1 will fall in a given partition. The number k can be equal to any of $1, 2, \ldots, n$. It is clear that when k is fixed, the number of such partitions is equal to $\binom{n-1}{k-1}\tau(n-k)$. In that case,

$$\tau(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} \tau(n-k).$$

It is easy to see that

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \tau(n-k) = \sum_{k=0}^{n-1} \binom{n-1}{k} \tau(k),$$

and so finally, we have shown that

$$\tau(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} \tau(k). \tag{7}$$

With the aid of formula (7) it isn't difficult to calculate that $\tau(4) = 15$, $\tau(5) = 52$, $\tau(6) = 203$, $\tau(7) = 877$, $\tau(8) = 4140$: As n increases, the numbers $\tau(n)$ grow quickly.

We'll use the recurrence formula (7) to prove a curious proposition in which the number e makes an unexpected appearance:

$$e \cdot \tau(n) = \frac{1^n}{1!} + \frac{2^n}{2!} + \cdots + \frac{k^n}{k!} + \cdots \quad (n \geq 1). \tag{8}$$

PROOF. Let us apply the method of mathematical induction. When $n = 1$, equation (8) holds by virtue of (5). Let us now assume that (8) is true for all nonnegative integers less than a certain n. Let us prove that in this case, it is

[3] Numbers of the type $\tau(n)$ are called *Bell numbers* (*Kvant* 1978, no. 7).

true for n as well. According to (5) and our induction hypothesis, the following equalities hold:

$$\begin{aligned} e\cdot\tau(0) &= 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}+\cdots, \\ e\cdot\tau(1) &= \frac{1}{1!}+\frac{2}{2!}+\cdots+\frac{k}{k!}+\cdots, \\ e\cdot\tau(2) &= \frac{1^2}{1!}+\frac{2^2}{2!}+\cdots+\frac{k^2}{k!}+\cdots, \\ &\cdots \\ e\cdot\tau(n-1) &= \frac{1^{n-1}}{1!}+\frac{2^{n-1}}{2!}+\cdots+\frac{k^{n-1}}{k!}+\cdots. \end{aligned}$$

(The infinite sums on the right-hand sides of these equations naturally must be understood as the limits of partial sums.) Let us add these equations term by term after multiplying the kth equation by $\binom{n-1}{k}$, $0\le k\le n-1$. On the left-hand side, according to (7), we obtain $e\cdot\tau(n)$; on the right-hand side we get

$$1+\frac{\varphi(1)}{1!}+\frac{\varphi(2)}{2!}+\cdots+\frac{\varphi(k)}{k!}+\cdots,$$

where

$$\varphi(t)=\binom{n-1}{0}+\binom{n-1}{1}t+\binom{n-1}{2}t^2+\cdots+\binom{n-1}{n-1}t^{n-1}=(1+t)^{n-1}.$$

It follows that

$$\begin{aligned} e\cdot\tau(n) &= 1+\frac{2^{n-1}}{1!}+\frac{3^{n-1}}{2!}+\frac{(k+1)^{n-1}}{k!}+\cdots \\ &= \frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\cdots+\frac{(k+1)^n}{(k+1)!}+\cdots, \end{aligned}$$

which is what we wanted to show.

From (8) we obtain the pretty formula

$$\tau(n)=\frac{1}{e}\sum_{k=1}^{\infty}\frac{k^n}{k!},$$

which is, however, inconvenient for computing $\tau(n)$. Below, we will derive direct (in the sense of nonrecurrence) finite (not the sum of an infinite series) formulas for calculating $\tau(n)$.

Let us call the *rank* of a given partition the number of classes of which it's composed. Let us denote by $c_{n,k}$ the number of partitions of rank k that an n-element set possesses.[4] It is evident that a rank can have values equal only to $1, 2, \ldots, n$. Therefore,

$$\tau(n)=\sum_{k=1}^{n}c_{n,k}. \tag{9}$$

For $k>n$ we define $c_{n,k}=0$.

[4]The numbers $c_{n,k}$ are called *Stirling numbers*.

It is easy to derive a recurrence relation for the numbers $c_{n,k}$. Suppose $M = \{a_1, a_2, \dots, a_n\}$, $M' = \{a_2, \dots, a_n\}$. Let us break down the partitions of the set M with rank k into two groups: those that contain the one-element set $\{a_1\}$ as a class of the partition and those that do not. It is clear that there are as many partitions of the first kind as there are partitions of the set M' of rank $k-1$, that is, $c_{n-1,k-1}$. And there are k times as many partitions of the second kind as there are partitions of the set M' of rank k (the element a_1 can be placed in any of the k classes of the partition of the set M'), that is, $k \cdot c_{n-1,k}$. Thus, for $n > 1,\ k > 1$,

$$c_{n,k} = c_{n-1,k-1} + k \cdot c_{n-1,k}. \tag{10}$$

Let us prove that for $n \geq 1,\ k \geq 1$,

$$c_{n,k} = \sum_{s=0}^{k-1} \frac{(-1)^s (k-s)^{n-1}}{s!(k-s-1)!}. \tag{11}$$

We shall let $b_{n,k}$ denote the right-hand side of (11). If $k = 1$, then $b_{n,1} = 1 = c_{n,1}$. If $k > 1$ and $n = 1$, then

$$\begin{aligned} b_{1,k} &= \sum_{s=0}^{k-1} \frac{(-1)^s}{s!(k-s-1)!} = \frac{1}{(k-1)!} \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s} \\ &= \frac{1}{(k-1)!}(1-1)^{k-1} = 0 = c_{1,k}. \end{aligned}$$

Finally, if $k > 1$ and $n > 1$, then

$$\begin{aligned} b_{n,k} &= \sum_{s=0}^{k-1} \frac{(-1)^s (k-s)^{n-1}}{s!(k-s-1)!} = \sum_{s=0}^{k-1} \frac{(-1)^s (k-s)^{n-2}(k-s)}{s!(k-s-1)!} \\ &= k \sum_{s=0}^{k-1} \frac{(-1)^s (k-s)^{n-2}}{s!(k-s-1)!} + \sum_{s=0}^{k-1} \frac{(-1)^{s+1}(k-s)^{n-2} s}{s!(k-s-1)!} \\ &= k \cdot b_{n-1,k} + \sum_{s=1}^{k-1} \frac{(-1)^{s+1}(k-s)^{n-2} s}{s!(k-s-1)!} \\ &= k \cdot b_{n-1,k} + \sum_{s=1}^{k-1} \frac{(-1)^{s+1}(k-s)^{n-2}}{(s-1)!(k-s-1)!} \\ &= k \cdot b_{n-1,k} + \sum_{s=0}^{k-2} \frac{(-1)^{s+2}(k-s-1)^{n-2}}{s!(k-s-2)!} \\ &= k \cdot b_{n-1,k} + \sum_{s=0}^{k-2} \frac{(-1)^{s}(k-s-1)^{n-2}}{s!(k-s-2)!} = k \cdot b_{n-1,k} + b_{n-1,k-1}. \end{aligned}$$

In this way, both numbers $c_{n,k}$ and $b_{n,k}$ satisfy the same recurrence relation of the type (10); furthermore, their "initial values" for $k = 1$ (and any $n \geq 1$) and $n = 1$ (and any $k \geq 1$) coincide. Consequently (Figure 2), $c_{n,k} = b_{n,k}$ for any $n \geq 1$ and $k \geq 1$.

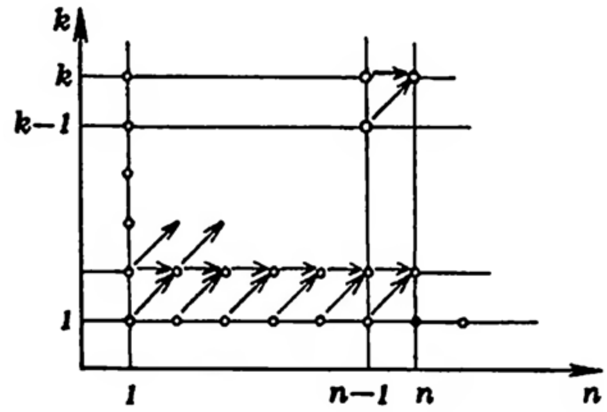

FIGURE 2

From (9) and (11) we have

$$\tau(n)=\sum_{k=1}^{n}\sum_{s=0}^{k-1}\frac{(-1)^s(k-s)^{n-1}}{s!(k-s-1)!}. \tag{12}$$

Let us set $k-s=t$. Then $1\le t\le n$. For a fixed t in the double sum (12) the value of s can vary from 0 to $n-t$. After collecting the coefficients for t^{n-1}, we can express formula (12) in the following way:

$$\tau(n)=\sum_{t=1}^{n}\Big(\sum_{s=0}^{n-t}\frac{(-1)^s}{s!}\Big)\frac{t^{n-1}}{(t-1)!}.$$

The numbers $c_{n,k}$, which have a clear combinatorial meaning, suddenly appear in a certain algebraic problem connected with polynomials.

Let us define $x^{(1)}=x$, $x^{(2)}=x(x-1)$, $x^{(3)}=x(x-1)(x-2)$, and so on. The polynomials $1, x^{(1)}, x^{(2)},\dots,x^{(n)}$ are, respectively, of degree $0,1,2,\dots,n$. Therefore, any polynomial whose degree in x is less than n can be expressed in only one way as the sum of these polynomials (with certain coefficients). In particular,

$$x^n=a_{n,1}x^{(1)}+a_{n,2}x^{(2)}+\cdots+a_{n,n}x^{(n)} \tag{13}$$

for some numbers $a_{n,k}$, $n\ge1$, $1\le k\le n$. For $k>n$, let us set $a_{n,k}=0$. It turns out that $a_{n,k}=c_{n,k}$. Let us prove this.

Assuming in (13) that $x=1$, we obtain $a_{n,1}=1=c_{n,1}$. If $k>1$ and $n=1$, then $a_{1,k}=0=c_{1,k}$. Now let $k>1$ and $n>1$. We have

$$\begin{aligned}
x^n&=\sum_{k=1}^{n}a_{n,k}x^{(k)}=x\cdot x^{n-1}=x\cdot\sum_{i=1}^{n-1}a_{n-1,i}x^{(i)}\\
&=\sum_{i=1}^{n-1}a_{n-1,i}x^{(i)}\big[(x-i)+i\big]=\sum_{i=1}^{n-1}a_{n-1,i}\big[x^{(i)}(x-i)+ix^{(i)}\big]\\
&=\sum_{i=1}^{n-1}a_{n-1,i}\Big[x^{(i+1)}+ix^{(i)}\Big]=\sum_{i=1}^{n-1}a_{n-1,i}x^{(i+1)}+\sum_{i=1}^{n-1}a_{n-1,i}ix^{(i)}.
\end{aligned}$$

Comparing the coefficients for $x^{(k)}$, we obtain $a_{n,k} = a_{n-1,k-1} + k \cdot a_{n-1,k}$. As above, from this it follows that $a_{n,k} = c_{n,k}$ for all n, k.

EXERCISE. Prove the following equality:

$$\tau(n-1) = \sum_{t=0}^{n-1} (-1)^t \binom{n-1}{t} \tau(n-t).$$

Addendum: Let us Calculate the Number e (by G. Sorokin)

Above we proved that the number

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

is equal to the sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots .$$

However, it is possible to come up with a series whose partial sums approach e much more quickly than the partial sums of this series. Below, we offer several exercises, which, if you solve them, will enable you—by means of some easy computations—to write down the decimal expansion of e to many decimal places.

EXERCISE 1. Prove that the following equality holds for $n \geq 2$:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{n \cdot n!} = 3 - \frac{1}{1 \cdot 2 \cdot 2!} - \frac{1}{2 \cdot 3 \cdot 3!} - \cdots - \frac{1}{(n-1) \cdot n \cdot n!}.$$

Hint: Apply the method of mathematical induction.

EXERCISE 2. Prove that

$$e = 3 - \sum_{n=0}^{\infty} \frac{1}{(n+1) \cdot (n+2) \cdot (n+2)!}.$$

EXERCISE 3. Prove that

$$e = \frac{87}{32} - \sum_{n=0}^{\infty} \frac{1}{(n+4) \cdot (n+5) \cdot (n+5)!}.$$

EXERCISE 4. Let

$$e = \frac{87}{32} - \sum_{n=0}^{k-1} \frac{1}{(n+4) \cdot (n+5) \cdot (n+5)!} - R_k.$$

Determine the remainder R_k.

EXERCISE 5. Calculate the approximate value of e using three elements of the series in Exercise 4 and determine the error in your approximation.

EXERCISE 6. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{(n+4)\cdot(n+5)\cdot(n+5)!} < 0.00047.$$

Translated by ILYA BERNSTEIN