Rational Approximations and Transcendence

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Let's begin this article with a couple of definitions.

A real number that is a root of a polynomial with integer coefficients is called *algebraic*. Real numbers that are not algebraic are called *transcendental*.

While there are infinitely many algebraic numbers and infinitely many transcendental numbers, "most" numbers are transcendental. This fact is usually proved in the following way. The set of algebraic numbers is countable, because the set of polynomials with integer coefficients is countable and each such polynomial has a finite number of roots. On the other hand, the set of all real numbers is uncountable.[1]

This proof is interesting because it not only establishes the existence of transcendental numbers, but also shows that in a certain sense there are more of them than of algebraic numbers. However, this proof also has a significant shortcoming: It isn't constructive. That is, it doesn't actually produce any particular nonalgebraic number. To be sure, everyone knows some examples of transcendental numbers: π and e, for instance. But proving the transcendence of these numbers is not easy at all. And in general, proving the transcendence of a specific number often turns out to be very complicated. For instance, in one of his famous problems, David Hilbert asked for a proof of the transcendence of the number $2^{\sqrt{2}}$. It was a long time before this problem finally succumbed to the efforts of the Soviet mathematician A. O. Gel'fond.

In this article we will indicate one of the methods for constructing transcendental numbers (with the proof of their transcendence). Our basic instrument will be the theory of rational approximations. In our articles about rational approximations[2] (we should mention that it isn't necessary to know the content of those articles in order to understand this one), we showed that the worst numbers to approximate using rational numbers are quite nice algebraic numbers such as $\sqrt{2}$, $(\sqrt{5}+1)/2$, and so on. We will give a precise definition of a "well-approximable" irrational number, and we will prove that all irrational numbers that can be well approximated using rational numbers are transcendental. This will allow us to construct as many examples of transcendental numbers as we wish.

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[1] An explanation of which sets are called countable and which sets are called uncountable can be found in the article by Yu. P. Lysov, "Which Numbers Are There More Of?" in *Kvant* 1973, no. 12, pp. 26–28. The proof of the uncountability of the set of real numbers is provided on p. 8 of the same issue of *Kvant*.

[2] *Kvant* 1971, nos. 6 and 11. (See pp. 27–47 in this book.)

DEFINITION. An irrational number α is called well-approximable if for all positive integers N, n, there is a rational number $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq^n}.$$

It is easy to construct as many well-approximable numbers as we wish. For example, let

$$\alpha = 0.1\underbrace{0\dots0}_{m_1-1}1\underbrace{0\dots0}_{m_2-1}1\underbrace{0\dots0}_{m_3-1}10\dots,$$

where $m_1, m_2, m_3, \dots$ is an increasing sequence of integers greater than 1. Let us set

$$\alpha_k = 0.1\underbrace{0\dots0}_{m_1-1}1\underbrace{0\dots0}_{m_2-1}1\dots1\underbrace{0\dots0}_{m_k-1}1.$$

It is evident that α_k is a rational number with denominator

$$q_k = 10^{m_1+\dots+m_k+1}$$

and that

$$|\alpha - \alpha_k| = \alpha - \alpha_k = 0.\underbrace{0 \quad \dots \quad 0}_{m_1+\dots+m_k+m_{k+1}}10\dots < 2 \cdot 10^{-m_1-\dots-m_k-m_{k+1}} < 10^{-m_{k+1}}.$$

Let us now assume that the sequence $m_1, m_2, m_3, \dots$ increases so rapidly that $m_{k+1} \geq k(m_1+\dots+m_k+2)$ for all k. (An example is the sequence $1^1, 2^2, 3^3, 4^4, \dots$. Prove it!) Then for all k,

$$\begin{aligned} \frac{1}{kq_k^k} &= \frac{10^{-k(m_1+\dots+m_k+1)}}{k} \\ &> 10^{-k}\,10^{-k(m_1+\dots+m_k+1)} \\ &= 10^{-k(m_1+\dots+m_k+2)} \\ &\geq 10^{-m_{k+1}} > |\alpha - \alpha_k|, \end{aligned}$$

and therefore if n, N are any positive integers and $k > n$ and $k > N$, then

$$|\alpha - \alpha_k| < \frac{1}{kq_k^k} < \frac{1}{Nq_k^n}.$$

Thus α is a well-approximable number.

Let us now prove the following theorem.

THEOREM (Joseph Liouville (1809–1882)). *A well-approximable number cannot be algebraic.*

PROOF. Let us take an irrational algebraic number α and show that it is not well-approximable. Since α is algebraic, there exists a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

with integer coefficients such that $P(\alpha) = 0$.

We will assume that $P(x)$ has no rational roots. Indeed, if $P(x)$ did have a rational root, say a, then $P(x)$ could be divided by $x - a$ with no remainder. The quotient polynomial would obviously have rational coefficients, so that multiplying it by an appropriate integer, we would get a polynomial with integer coefficients for which α would remain a root and whose degree would be one less than that of the initial polynomial. Repeating this procedure as many times as necessary, we would eventually obtain a polynomial with integer coefficients and with the root α but with no rational roots.

Let us call A the greatest of the numbers $|a_0|, |a_1|, \dots, |a_n|$ and set $B = |\alpha| + 1$, $N = n^2 A B^{n-1}$.

Let us now show that whatever the fraction $\frac{p}{q}$ may be, the following inequality holds:

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{Nq^n}, \tag{$*$}$$

and it is this inequality from which it follows that the number α is not well-approximable.

Let $\frac{p}{q}$ be any fraction. If

$$\left|\alpha - \frac{p}{q}\right| > 1,$$

then inequality $(*)$ is obviously satisfied, so we can limit ourselves to cases where

$$\left|\alpha - \frac{p}{q}\right| \leq 1.$$

In particular (and this is very important),

$$\left|\frac{p}{q}\right| \leq |\alpha| + 1 = B.$$

From what was said above it follows that $P\left(\frac{p}{q}\right) \neq 0$.

Since

$$q^n \left[a_0 \left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + \cdots + a_n\right]$$

is an integer not equal to zero, it follows that

$$\left|q^n \left[a_0 \left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + \cdots + a_n\right]\right| \geq 1,$$

and consequently,

$$\left|a_0 \left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + \cdots + a_n\right| \geq \frac{1}{q^n}.$$

On the other hand, taking into account that

$$a_0\alpha^n + a_1\alpha^{n-1} + \cdots + \alpha_n = 0,$$

we obtain

$$\begin{aligned}
&\left|a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_n\right| \\
&\qquad = \left|a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_n - (a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_n)\right| \\
&\qquad \le |a_0|\left|\left(\frac{p}{q}\right)^n - \alpha^n\right| + |a_1|\left|\left(\frac{p}{q}\right)^{n-1} - \alpha^{n-1}\right| + \cdots + |a_{n-1}|\left|\frac{p}{q} - \alpha\right| \\
&\qquad \le A\left|\left(\frac{p}{q}\right)^n - \alpha^n\right| + A\left|\left(\frac{p}{q}\right)^{n-1} - \alpha^{n-1}\right| + \cdots + A\left|\frac{p}{q} - \alpha\right| \\
&\qquad = A\left|\frac{p}{q} - \alpha\right| \left\{ \left|\left(\frac{p}{q}\right)^{n-1} + \left(\frac{p}{q}\right)^{n-2}\alpha + \cdots + \left(\frac{p}{q}\right)\alpha^{n-2} + \alpha^{n-1}\right| \right. \\
&\qquad\qquad + \left|\left(\frac{p}{q}\right)^{n-2} + \left(\frac{p}{q}\right)^{n-3}\alpha + \cdots + \left(\frac{p}{q}\right)\alpha^{n-3} + \alpha^{n-2}\right| \\
&\qquad\qquad \left. + \cdots + \left|\frac{p}{q} + \alpha\right| + 1 \right\} \\
&\qquad \le A\left|\frac{p}{q} - \alpha\right| \left\{nB^{n-1} + (n-1)B^{n-2} + \cdots + 2B + 1\right\} \\
&\qquad \le A\left|\frac{p}{q} - \alpha\right| n \cdot nB^{n-1} = n^2AB^{n-1}\left|\frac{p}{q} - \alpha\right| = N\left|\frac{p}{q} - \alpha\right|.
\end{aligned}$$

(We have made use of the fact that both α and $\left|\frac{p}{q}\right|$ are less than B.)

Therefore,

$$N\left|\frac{p}{q} - \alpha\right| \ge \frac{1}{q^n};$$

that is,

$$\left|\alpha - \frac{p}{q}\right| \ge \frac{1}{Nq^n},$$

and the theorem is proved. □

Thus all well-approximable numbers (including the number α mentioned earlier) are transcendental.

In conclusion, let us note that the theorem we have proved by no means exhaustively reflects the connection between the algebraicity of numbers and the nature of their rational approximations. Further development of these methods will allow us to prove the transcendence of a variety of different numbers (by the way, the transcendence of the number e is usually proved along the same lines). It should be noted that research in this area continues even to this day. One of the most striking results in this field in recent years has been the work of the mathematician Klaus Friedrich Roth, who was awarded the prestigious Fields Medal at the International Congress of Mathematicians in Edinburgh in 1958.

The theorem proved by Roth states that if α is an algebraic number, then there exist only finitely many fractions $\frac{p}{q}$ such that for any given $\epsilon > 0$,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

Therefore, our theorem about well-approximable numbers would have remained true if in our definition of a well-approximable number we had demanded the existence of the fraction $\frac{p}{q}$ with

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq^n}$$

not for all positive n and N, but for only a single n strictly greater than 2.

By the way, it is impossible to get rid of the ϵ in Roth's theorem. This follows from the Hurwitz–Borel theorem, which was proved in our article "On Best Approximations. II" (*Kvant*, 1971, no. 11; see pp. 37–47 in this book).

Translated by ILYA BERNSTEIN