CHAPTER I

DIFFERENTIATION

LEBESGUE'S THEOREM ON THE DERIVATIVE OF A MONOTONIC FUNCTION

1. Example of a Nondifferentiable Continuous Function

In classical analysis it is generally assumed that the functions considered possess derivatives, indeed even continuous derivatives up to some order, although it is true that occasional exceptions are allowed. In spite of this, up to the beginning of the present century mathematicians rarely asked whether the functions belonging to a particular category, for example the continuous or the monotonic functions, necessarily possess derivatives; or, if they are known not to possess derivatives everywhere, then at least whether they possess them on the complement of a set whose nature can be made precise. The results in this direction were limited to several almost obvious facts, for example that a convex function necessarily has a left and a right derivative at each point and therefore is differentiable at every value of x except at most for a denumerable set of exceptional values.

The first serious consideration of these problems came in 1806, when in a paper entitled "Sur la théorie des fonctions dérivées" the great scholar AMPÈRE [**1**][1] tried without success to establish the differentiability of an "arbitrary" function except at certain "particular and isolated" values of the variable. Taking into account the evolution of the idea of function one is led to believe—although the original text says nothing positive on this point—that the efforts of Ampère could hardly have been directed beyond functions consisting of monotonic components.

During the entire nineteenth century, as fruitful as this century was for the development of analysis, the solution of the problem did not advance; it even seems at first glance that mathematicians were farther from it. In fact, the first important result, after more than half a century, was furnished by the critique of WEIERSTRASS,[2] who put an end to the repeated attempts

[1] The numbers and asterisks in square brackets refer to the bibliography at the end of the book.

[2] Published by DU BOIS-REYMOND [**1**].

to establish the differentiability of an arbitrary continuous function by constructing *a continuous function without a derivative.* Afterwards these examples multiplied, simpler and simpler ones were invented, and their mutual relationships and relations to other problems were carefully investigated. This was a work in which almost all the great masters of analysis of the second half of the century took part and which has continued up to our time. Here is such an example, perhaps the most elementary, which is due to VAN DER WAERDEN [**1**] and is based on the obvious fact that an infinite sequence of integers can be convergent only if its terms remain equal from some point on.

Let us agree to say, as usual, that the function $f(x)$ possesses a derivative at the point x when the ratio

$$\frac{f(x+h)-f(x)}{h}$$

tends to a finite limit as $h \to 0$ and $x + h$ runs through values for which $f(x + h)$ has a meaning. Denote by $\{x\}$ the distance from x to the nearest integer. Let us form the function

$$f(x) = \sum_{n=0}^{\infty} \frac{\{10^n x\}}{10^n}; \tag{1}$$

since the terms of this series represent continuous functions and, furthermore, since the series is majorized by the geometric series $\Sigma\, 10^{-n}$, the function $f(x)$ is obviously continuous. But in trying to calculate the derivative at the point x we shall encounter a contradiction.

Let us observe that we can obviously restrict ourselves to the case where $0 \leq x < 1$ and let us write x in the form

$$x = 0.a_1\, a_2 \ldots a_n \ldots,$$

with the agreement that when the option arises we shall write x in the form of a finite decimal fraction completed with zeros. We distinguish two cases according as

$$0.a_{n+1}\, a_{n+2} \ldots \leq \tfrac{1}{2} \text{ or } > \tfrac{1}{2}.$$

In the first case,

$$\{10^n x\} = 0.a_{n+1}\, a_{n+2} \ldots,$$

whereas in the second,

$$\{10^n x\} = 1 - 0.a_{n+1}\, a_{n+2} \ldots.$$

We set $h_m = -10^{-m}$ when a_m equals 4 or 9 and $h_m = 10^{-m}$ otherwise. Consider the ratio

$$\frac{f(x+h_m)-f(x)}{h_m}; \tag{2}$$

by formula (1) this ratio can be expressed by a series of the form

$$10^m \sum_{n=0}^{\infty} \pm \frac{\{10^n (x \pm 10^{-m})\} - \{10^n x\}}{10^n}$$

But it is clear that the numerators are zero starting with $n = m$ and, on the other hand, that for $n < m$ they reduce to $\pm 10^{n-m}$; therefore the corresponding terms of our expression equal ± 1, and consequently the value of ratio (2) is an integer which may or may not be positive, but in any case is even or odd according to the parity of $m - 1$. Hence the sequence of ratios (2), since it is formed of integers of varying parity, cannot converge.

2. Lebesgue's Theorem on the Differentiation of a Monotonic Function. Sets of Measure Zero

We consider next the class of monotonic functions. We owe to Lebesgue the following theorem, one of the most striking and most important in real variable theory.

THEOREM. *Every monotonic function $f(x)$ possesses a finite derivative at every point x with the possible exception of the points x of a set of measure zero, or, as it is often phrased, almost everywhere.*

Before defining the expressions used, let us add that Lebesgue established his theorem using *the additional hypothesis of the continuity of $f(x)$*. He did this in 1904 in the first edition of his book on integration [*], and it appeared at the end of the last chapter as the final result of the entire theory. However, neither the idea of integral nor that of measure appear in the statement of the theorem. In fact, the idea of a set of measure zero does not depend essentially on the general theory of measure, and the main properties of these sets can be established in a few words.

According to Lebesgue, *a set of measure zero* is a set of values x which can be covered by a finite number or by a denumerable sequence of intervals whose total length (i. e., the sum of the individual lengths) is arbitrarily small. It follows immediately from this definition that every subset of such a set is also of measure zero. The same is true for the union of a finite number or of a denumerable sequence of such sets; in fact, we have only to cover these sets respectively by a system of intervals whose total length does not exceed $\frac{\varepsilon}{2^n}$; then the total length of all these intervals covering the union of our sets will not exceed the quantity ε. In particular, every finite or denumerable set of values of x is of measure zero.

It will sometimes be advantageous to give this definition the following form. A set E is of measure zero if it can be covered by a sequence of intervals