Concentration of the chromatic number of sparse random graphs

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Context and Main Question

- **Random graph** $G_{n,p}$: n-vertex graph where each of $\binom{n}{2}$ possible edges included independently with probability p
- **Chromatic number** $\chi(G)$: minimum number of colors needed to color vertices of G s.t. no two adjacent vertices have same color

Main Question

Suppose there is an interval of length $\ell(n, p)$ that contains chromatic number $\chi(G_{n,p})$ with high probability. How small can $\ell(n, p)$ be?

Past Results: constant p

Bollobas 1988

For constant edge-probability $p \in (0, 1)$, whp

$$\chi(G_{n,p}) = (1 + o(1)) \frac{n}{2 \log_{1/(1-p)} n}.$$

So $\ell(n, p) = o(n / \log_b n)$.

- **Lower bound**: Show largest ISET is of size $(2 + o(1)) \log_{1/(1-p)} n$.
- **Upper bound**: Repeatedly pull out ISET of size $2 \log_{1/(1-p)} n$ until $O(\sqrt{n}/ \log n)$ vertices are left (via Janson's inequality).

Past Results: all p

Shamir and Spencer 1987

- $\ell(n,p) \leq \omega\sqrt{n}$ for any $p(n)$,
- $\ell(n,p) \leq \omega\sqrt{np}\log n$ for $p = n^{-\alpha}, \alpha \in (0, 1/2)$

- **Basic-Idea**: Via Martingale argument to show that whp there exists $\Lambda \geq 0, Z \subseteq V, |Z| \leq \omega\sqrt{n}$

$$\Lambda \leq \chi(G_{n,p}) \leq \Lambda + \chi(G_{n,p}[Z])$$

- For $p = n^{-\alpha}$, easy to remove extra $\log n$ term with modern argument
- **Key-Task**: argue that $G_{n,p}[Z]$ is sparse

Past Results

log improvement by Alon (and later independently by Scott)

For $p \in [0,1]$ constant, $\ell(n,p) \le \omega\sqrt{n}/\log n$

- **Idea**: Repeatedly remove ISET of size $\Theta(\log n)$ from $G_{n,p}[Z]$
- If we use Janson's inequality to pull out the ISET, this only works until $p = n^{-\alpha}$ for some small $\alpha > 0$

Alon and Krivelevich

Let $\epsilon > 0$. If $p \le n^{-1/2-\epsilon}$, then $\ell(n,p) \le 2$

New result: Sparse case $p = o(1)$

Surya and Warnke (2022+)

Let $\epsilon > 0$. If $p \geq n^{-1/2+\epsilon}$, then

$$\ell(n, p) \leq \frac{\omega\sqrt{np}}{\log n}$$

- Use *density argument* instead of large deviation inequalities

More detailed statement: Sparse case $p = o(1)$

Surya and Warnke (2022+)

- If $\omega\sqrt{np} \gg \log n$, then

$$\ell(n,p) = O\left(\frac{\omega\sqrt{np}}{\log(\omega\sqrt{np}/\log n)}\right)$$

- If $\omega\sqrt{np} \ll \log n$, then

$$\ell(n,p) = O\left(\frac{\log n}{\log(\log n/(\omega\sqrt{np}))}\right)$$

- If $p = n^{-\alpha}$, $\alpha \in (0, 1/2)$ we have $\ell(n,p) = O\left(\frac{\omega\sqrt{np}}{\log n}\right)$, extending log improvement of Alon.
- Match the best known upper bound up to some constant factor when p constant and $p \leq n^{-1/2-\epsilon}$

Key Ingredient: Greedy Algorithm

Will focus on controlling $\chi(G_{n,p}[Z])$.

We use greedy algorithm in two ways, *exploiting small degree vertices*:

- Pull out largest independent sets until $O(\frac{\log n}{p})$ vertices are left, which will have typical size $\simeq O(\log(\omega\sqrt{n}p/\log n)/p)$.
  - *Refined analysis*: as fewer vertices remain, the independent sets get smaller (exploit that few vertices remain).
- Pick the minimum degree vertex among the remaining vertices, which will have degree $O(\log n)$.

Chernoff bound + Union bound: small degree conditions holds whp

Greedy Lemmas

To iteratively pull out largest independent set (until few vertices remain):

Large independent sets: greedy bound

Given graph G and $0 < d < 1 < u$ with $\delta(G[S]) \leq d(|S|-1)$ for all $S \subseteq V(G)$ of size $|S| \geq u$. Then

$$\alpha(G[W]) \geq -\log_{(1-d)(1-1/u)}(|W|/u)$$

for any $W \subseteq V(G)$ of size $|W| \geq u$.

To color the remaining $O(\log n/p)$ vertices:

Chromatic number: greedy bound

Given a graph G with $\delta(G[S]) \leq r$ for all $S \subseteq V(G)$. Then

$$\chi(G) \leq r+1$$

Large independent sets: greedy bound

Given graph G and $0 < d < 1 < u$ with $\delta(G[S]) \leq d(|S| - 1)$ for all $S \subseteq V(G)$ of size $|S| \geq u$. Then $\alpha(G[W]) \geq -\log_{(1-d)(1-1/u)}\big(|W|/u\big)$ for any $W \subseteq V(G)$ of size $|W| \geq u$.

Construct independent set greedily: set $W_0 = W$ and, for $i \geq 1$, pick $w_i \in W_{i-1}$ with minimal degree in $G[W_{i-1}]$ and set

$$W_i = \big\{v \in W_{i-1} \ : \ v \text{ not adjacent to } w_i\big\}.$$

If $|W_{i-1}| \geq u$ holds, then $\deg_{G[W_i]}(w_i) \leq d(|W_i| - 1)$, implying that

$$|W_i| \geq (1-d)(|W_{i-1}| - 1) \geq (1-d)(1-1/u)|W_{i-1}|.$$

Large independent sets: greedy bound

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So W_i is non-empty for

$$i - 1 \leq -\log_{(1-d)(1-1/u)}\big(|W|/u\big) =: I\big(|W|\big),$$

so we terminate with an independent set $\{w_1, \ldots, w_j\} \subseteq W$ of size $j \geq \big\lfloor I(|W|) + 1\big\rfloor \geq I(|W|)$.

Very dense case $1 - p = n^{-\Omega(1)}$

- **Heuristic**: Optimal colouring is obtained by taking as many disjoint α-ISETs as possible, then covering the rest with $(\alpha - 1)$-ISETs
- **Main source of fluctuation**: *number of α-ISETs*

Conjecture

$(\log n)^{1/\binom{r}{2}} n^{-2/r} \ll 1 - p \ll n^{-2/(r+1)}$ for some integer $r \geq 1$. Let $\mu_{r+1} = \mu_{r+1}(n, p) := \binom{n}{r+1}(1-p)^{\binom{r+1}{2}}$ be the expected number of $r + 1$-ISET. Then

$$\ell(n, p) = \omega\sqrt{\mu_{r+1}}$$

Very dense case $1 - p = n^{-\Omega(1)}$ conjecture

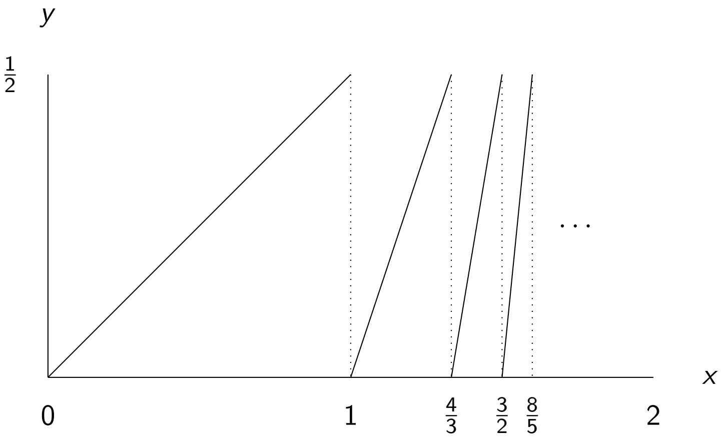

Figure: Conjecture predicts if $n^2(1-p) = n^{x+o(1)}$, then $\ell(n,p) = n^{y+o(1)}$

Concentration result: $1 - p = O(1/n)$

- **Number of ISET of size ≥ 3 is negligible**:
  Problem reduces to studying maximum matching on complement
- **Main source of fluctuation in maximum matching on $G_{n,q}$**:
  Fluctuation of isolated edges.

Theorem Surya and Warnke (2022+)

$C n \sqrt{q} \leq \ell(n,p) \leq \omega n \sqrt{q}$

- **Lower bound**: from fluctuation of isolated edges in complement $G_{n,q}$
- **Upper bound**: from Talagrand's inequality