The Density of Costas Arrays Decays Exponentially

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Objects of Interest:

"Costas Arrays" $\hat{=}$ permutation matrices $\underbrace{\text{with certain constraints}}_{\text{arising in Engineering Applications (Radar, Sonar, —)}}$

Question: How many "Costas Arrays" are there?

This Talk: Resolve "Exponential Decay" Conjecture:

$$\frac{\#\ n\times n \text{ Costas Arrays}}{\#\ n\times n \text{ Permutation Matrices}} \leq e^{-cn} \quad \text{for some } c>0$$

Motivation: Why should we care?

- Natural Enumerative Question
- Interest from Engineering / Applications
- Testbed for probabilistic techniques

Costas Arrays (J. Costas 1960s ; Navy Research)

Permutation Matrix where <u>all</u> vectors between different ones are distinct

Example:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$

Non-Examples:

$$\begin{bmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \end{bmatrix} \quad \begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

How used?

To determine transmit Frequencies & Times

Frequency

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$

Time

Key Combinatorial Property:

No shift of Time $(\Leftrightarrow)$ and Frequency $(\Updownarrow)$ can make any two different ones coincide

Quantity of Interest: (This Talk)

$C(n) = \#$ $n \times n$ Costas-Arrays

Background: Asymptotic Bounds

$C(n)$ = # $n \times n$ Costas-Arrays (≙ Permutation Matrices with Constraints)

Algebraic constructions:

Golomb-Taylor 1984

Gilbert 1964

$C(n) \geq 1$ for ∞ many n (e.g. n = prime − 1)

in fact $C(n) \geq n^{2-\varepsilon}$

Second Moment Method:

Benjamin Weiss 1984/85

Victor Reiner 1986

Huw Davies 1989

$$\frac{C(n)}{n!} \leq O\left(\frac{1}{n}\right)$$

"Polynomial Decay"

resolved one of 10 Fundamental OPEN PROBLEMS (Golomb-Taylor List 1984)

Conjecture:

Folklore late 1980s

Drakakis 2011

$$\frac{C(n)}{n!} \leq e^{-\Omega(n)}$$

"Exponential Decay"

Core Theoretical Problem (2011 Update of Golomb-Taylor List)

$C(n) = \#\ n\times n$ Costas-Arrays ($\hat{=}$ Permutation Matrices where all vectors between different ones are distinct)

Main Result (Correl, Swanson, W. 2021+)

Exponential decay: $\frac{C(n)}{n!} \leq e^{-cn}$ for all $n \geq 3$

- Verifies Core Problem + Speculations From 1980s
- Nearly best possible:
  - $\frac{C(n)}{n!} \geq \frac{1}{n!} \geq n^{-n} = e^{-n\log n}$ for ∞ many n
  - $n \geq 3$ needed: $\frac{C(n)}{n!} = 1$ for $n \in \{1,2\}$
- Proof: "Disjoint Approximation" Idea from Random Graph Theory. (to exploit Bounded Differences Inequality For Random Permutations)

Proof-Idea

① Find Forbidden "combinatorial" configuration:

Costas Arrays do <u>not</u> contain $\underbrace{\text{three equally spaced co-linear ones}}_{X = \#\text{ of these configurations in Random } n\times n \text{ Permutation Matrix}}$:

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

② Exploit Probabilistic Interpretation:

$$\frac{C(n)}{n!} = \mathbb{P}(\text{Random } n\times n \text{ Perm. Matrix is Costas Array}) \leq \mathbb{P}(X=0)$$

③ Disjoint Approximation + Bounded Differences Inequality

$$\frac{C(n)}{n!} \leq \mathbb{P}(X=0) \leq \mathbb{P}(X_D=0) \leq \cdots \leq e^{-\theta(n)}$$

max size of collection of three-one configurations counted by X that use <u>disjoint</u> ones

Open Problem: More precise estimate of $\frac{C(n)}{n!}$ or $C(n)$

$C(n)$ = # $n \times n$ Permutation Matrices where all vectors between different ones are distinct.

We Know: $n^{-n} \leq \frac{C(n)}{n!} \leq e^{-cn}$ for ∞ many n

Difficulty: Forbidden Configurations are non-standard.

"more geometric than usual"

$$\begin{bmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \end{bmatrix} \quad \begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Your Ideas?

- Enumerative / Analytic Combinatorics
- Probability / Concentration / Probabilistic Combinatorics
- Ad-Hoc Counting á la Stanley-Wilf Conjecture

Summary:

$C(n)$ = # $n \times n$ Costas-Arrays ($\hat{=}$ Permutation Matrices with Constraints)

Density of Costas Arrays

Exponential decay: $\frac{C(n)}{n!} \leq e^{-cn}$ for all $n \geq 3$

- Verifies Core Problem + Speculation from 1980s
- Proof: "Disjoint Approximation" Idea from Random Graph Theory.

Questions:

- Better bounds on $C(n)/n!$ or $C(n)$?
- What approaches might give $\frac{C(n)}{n!} \leq e^{-\omega(n)}$?