The degree-restricted random process is far from uniform

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Context and Overview

Random Graph Model: Random d-process

- Start with an empty graph on n vertices
- In each step: add one random edge so that max-degree stays $\leq d$

- Natural *random greedy algorithm to generate d-regular graph* (Balińska–Quintas 1985, Ruciński–Wormald 1992)

Basic Question: Wormald (1999)

How similar are d-process and uniform random d-regular graph G_d?

- Wormald conjectured they are similar (contiguous)

This Talk: Variant for degree sequences $\mathbf{d_n}$

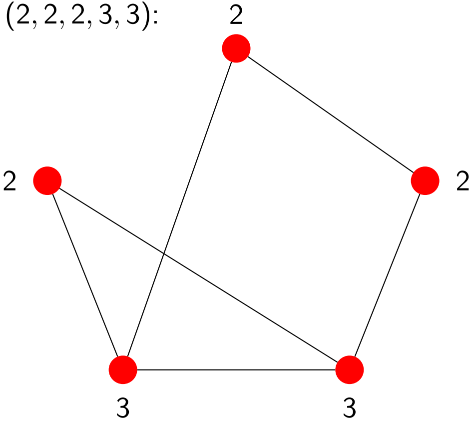
Degree-restricted process <u>differs</u> from uniform $G_{\mathbf{d_n}}$ for *irregular* $\mathbf{d_n}$

Variant for degree sequences $\mathbf{d_n} = (d_1, \dots, d_n)$

Degree-restricted random $\mathbf{d_n}$-process

- Start with an empty graph on n vertices
- In each step: add one random edge to the graph,
  so that *the degree of each vertex* v_i *stays* $\leq d_i$

Example for $\mathbf{d_5} = (2, 2, 2, 3, 3)$:



Variant for degree sequences $\mathbf{d_n} = (d_1, \ldots, d_n)$

Degree-restricted random $\mathbf{d_n}$-process

- Start with an empty graph on n vertices
- In each step: add one random edge to the graph,
  so that *the degree of each vertex* v_i *stays* $\leq d_i$

Basic Distributional Question:

How similar is final graph $G^P_{\mathbf{d_n}}$ of degree-restricted random $\mathbf{d_n}$-process to a uniform random graph $G_{\mathbf{d_n}}$ with degree sequence $\mathbf{d_n}$?

- **Statistics:** can we (algorithmically) distinguish them?
- **Combinatorial Probability:** do both have similar typical properties?
- **Algorithms:** can $\mathbf{d_n}$-process be used for random sampling?
- **Modeling/Physics:** does the simplest model work?

Main Result: $\mathbf{d_n}$-process and uniform model differ

$\mathbf{d_n} = (d_1, \ldots, d_n)$ *not nearly regular* : no degree appears $\geq 0.99n$ times

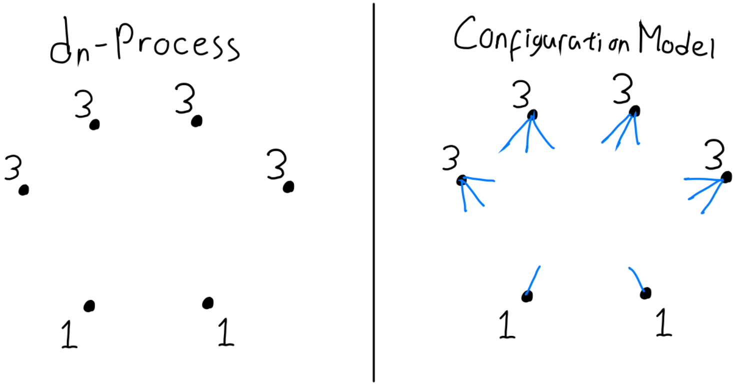
Molloy, Surya, Warnke (2022+)

If the bounded degree sequence $\mathbf{d_n}$ is *not nearly regular*, then can whp distinguish $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$ and uniform random $\mathbf{d_n}$-graph $G_{\mathbf{d_n}}$

Simple case (today): Assume # degree 1 vertices $\in [0.01n, 0.99n]$

- **Proof Idea:** *Show discrepancy in edge statistic*
  - Number of 1-1 edges differ whp (i.e., evolution of process matters)
- **Proof Technique:** *'Switching method'* applied to $\mathbf{d_n}$-process
  - Usually only applied to uniform models (not stochastic processes)

Intuition: why $\mathbf{d_n}$-process prefers 1-1 edges



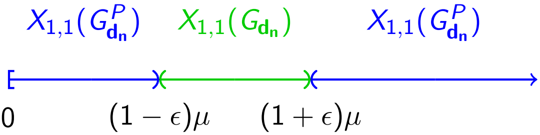
Main Technical Result: Discrepancy in Edge Statistic

$X_{1,1}(G) = \#$ of edges with endpoints of degree 1 in G

Can distinguish both models via $X_{1,1}$

There exists μ and $\epsilon = \epsilon(\Delta) > 0$ such that with high probability

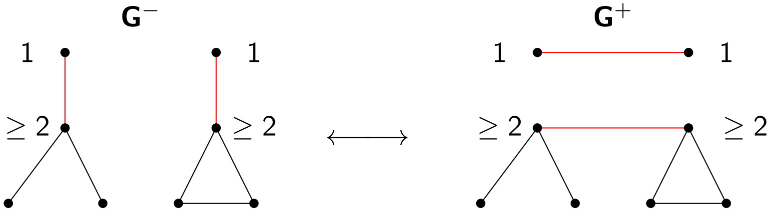
$$X_{1,1}(G_{\mathbf{d_n}}) \in [(1-\epsilon)\mu, (1+\epsilon)\mu] \quad \text{and} \quad X_{1,1}(G^P_{\mathbf{d_n}}) \notin [(1-\epsilon)\mu, (1+\epsilon)\mu]$$



- **Concentration of** $X_{1,1}(G_{\mathbf{d_n}})$: standard via configuration model
- **Understanding** $X_{1,1}(G^P_{\mathbf{d_n}})$: adapt *switching method* ($\longrightarrow$ This talk)

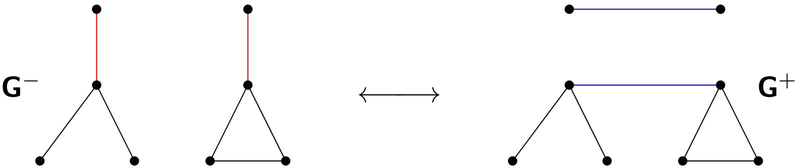
Switching: Change # of 1-1 edges by exactly one

Definition via Example:



- **Goal:** compare ratio $\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)/\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)$
  - # of 1-1 edges in G^+ and G^- differ by exactly one
  - switching between G^+ and G^- is 'local perturbation'
- **Extra difficulty for stochastic processes:**
  - no longer uniform (order of edges matters)
- **Solution**:
  - look at all trajectories (= edge orderings) yielding a graph

How Switching Affect $\mathbf{d_n}$-process Probabilities



Switching Lemma (for probabilities)

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)}{\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)} \geq 1 + \epsilon' \qquad \text{where } \epsilon' > 0 \text{ depends on } \Delta$$

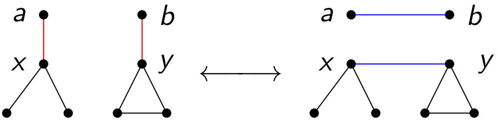
Proof Ideas:

- Expand probability based on edge-sequence σ of G

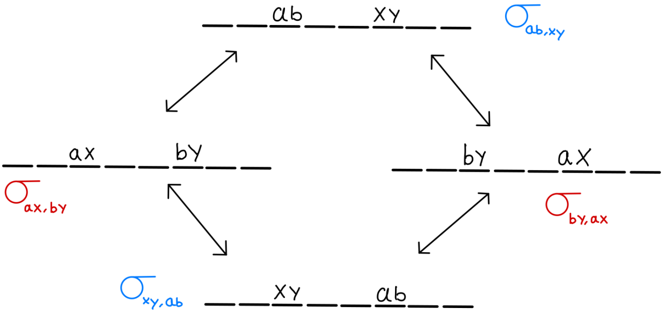
$$\mathbb{P}(G^P_{\mathbf{d_n}} = G) = \sum_{\sigma} \mathbb{P}(\mathbf{d_n}\text{-process returns } \sigma) =: \sum_{\sigma} \mathbb{P}(\sigma)$$

- *Understand how switching affects* $\mathbb{P}(\sigma)$
  - Compare (averaged ratios of) probabilities of similar trajectories

Switching edge-sequence



Edge-sequence σ: $\underline{e_1}$ $\underline{e_2}$ $\underline{e_3}$ $\underline{e_4}$ . . .

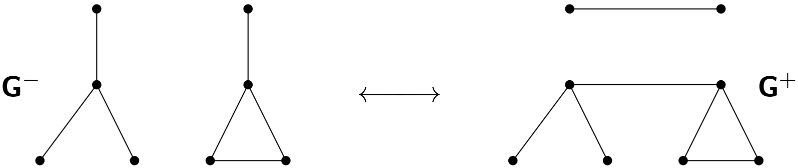


- **Key Inequality:**

$$\mathbb{P}(\sigma_{ab,xy}) + \mathbb{P}(\sigma_{xy,ab}) \geq \mathbb{P}(\sigma_{ax,by}) + \mathbb{P}(\sigma_{by,ax})$$

- LHS has one more 1-1 edge than RHS:
  - Indicates $\mathbf{d_n}$-process prefers more 1-1 edges

How Switching Affect $\mathbf{d_n}$-process Probabilities



Switching Lemma (for probabilities)

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)}{\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)} \geq 1 + \epsilon' \qquad \text{where } \epsilon' > 0 \text{ depends on } \Delta$$

Proof Idea: Use key inequality for all edge-sequences $\sigma = \sigma_{ab,xy}$ of G^+:

$$\mathbb{P}(G^P_{\mathbf{d_n}} = G^+) = \sum_{\sigma_{ab,xy}} \Big[\mathbb{P}(\sigma_{ab,xy}) + \mathbb{P}(\sigma_{xy,ab})\Big]$$

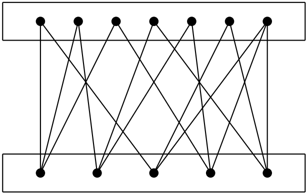
$$\geq \sum_{\sigma_{ax,by}} \Big[\mathbb{P}(\sigma_{ax,by}) + \mathbb{P}(\sigma_{by,ax})\Big] = \mathbb{P}(G^P_{\mathbf{d_n}} = G^-)$$

- Often win a factor of $1 + \epsilon$ in key inequality: get $1 + \epsilon'$

Switching: Graph Count Based on $X_{1,1}$

Notation: $G \in \mathbf{d_n}$ if G has degree sequence $\mathbf{d_n}$

Auxiliary Graph: by adding edge between G^+, G^-:



$\longrightarrow G_{\ell+1} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell + 1\}$

$\longrightarrow G_{\ell} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell\}$

Key Point: Auxiliary graph is roughly regular when $\ell \approx \mu$

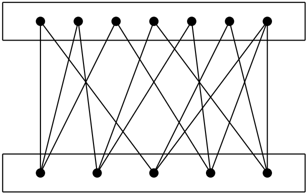
Switching lemma then implies:

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell+1})}{\mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell})} \geq 1 + \epsilon'$$

Uniform random graphs: switching easy

Notation: $G \in \mathbf{d_n}$ if G has degree sequence $\mathbf{d_n}$

Auxiliary Graph: by adding edge between G^+, G^-:



$\longrightarrow G_{\ell+1} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell + 1\}$

$\longrightarrow G_{\ell} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell\}$

Uniform random graph $G_{\mathbf{d_n}}$ simpler: classical switching works

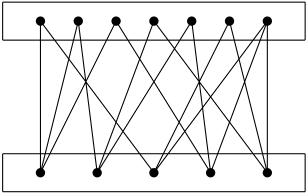
Crux is that normalization constants cancel out:

$$\frac{\mathbb{P}(G_{\mathbf{d_n}} \in G_{\ell+1})}{\mathbb{P}(G_{\mathbf{d_n}} \in G_{\ell})} = \frac{|G_{\ell+1}|}{|G_{\ell}|}$$

Degree-restricted process: why new ideas needed

Notation: $G \in \mathbf{d_n}$ if G has degree sequence $\mathbf{d_n}$

Auxiliary Graph: by adding edge between G^+, G^-:



$\longrightarrow G_{\ell+1} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell + 1\}$

$\longrightarrow G_{\ell} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell\}$

Degree-restricted random $\mathbf{d_n}$-process: why more complicated

Normalization constants *do not* cancel out:

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell+1})}{\mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell})} = \frac{\sum_{F \in G_{\ell+1}} \mathbb{P}(G^P_{\mathbf{d_n}} = F)}{\sum_{H \in G_{\ell}} \mathbb{P}(G^P_{\mathbf{d_n}} = H)}$$

Proof of Main Theorem (Sketch)

Definition: $\mathcal{N}_z = \{G \in \mathbf{d_n} : |X_{1,1}(G) - \mu| \leq z\}$

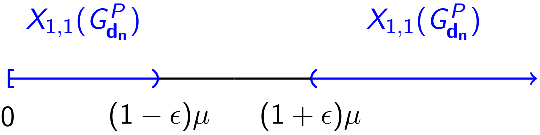
Key Point implies (for $z \leq 2\epsilon\mu$)

$$\frac{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_z]}{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_{z+1}]} \leq \frac{\sum_{\mu-z\leq\ell\leq\mu+z} \mathbb{P}(G^P_{\mathbf{d_n}} \in G_\ell)}{\sum_{\mu-z\leq\ell\leq\mu+z} \mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell+1})} \leq \frac{1}{1+\epsilon'}$$

Get exponential decay by telescoping product argument:

$$\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_{\epsilon\mu}) \leq \frac{\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_{\epsilon\mu})}{\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_{2\epsilon\mu})} = \prod_{z=\epsilon\mu}^{2\epsilon\mu-1} \frac{\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_z)}{\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_{z+1})} \leq \frac{1}{(1+\epsilon')^{\epsilon\mu}} \to 0$$

Conclusion: *whp number of 1-1 edges satisfies*



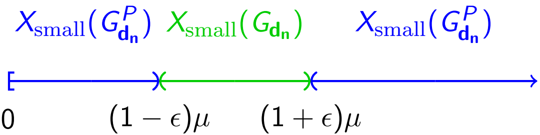
General case: more complicated

- **Small vertex**: $|\{v : \deg(v) \le s\}| \in [0.01n, 0.99n]$ (previously $s = 1$)
- **Small edge**: edge whose endpoints are small
- $X_{\text{small}}(G)$ = number of small edges in G

Goal: Distinguish both models via X_{small}

There exists μ and $\epsilon = \epsilon(\Delta) > 0$ such that with high probability

$$X_{\text{small}}(G_{\mathbf{d_n}}) \in [(1-\epsilon)\mu, (1+\epsilon)\mu] \quad \text{and} \quad X_{\text{small}}(G^P_{\mathbf{d_n}}) \notin [(1-\epsilon)\mu, (1+\epsilon)\mu]$$



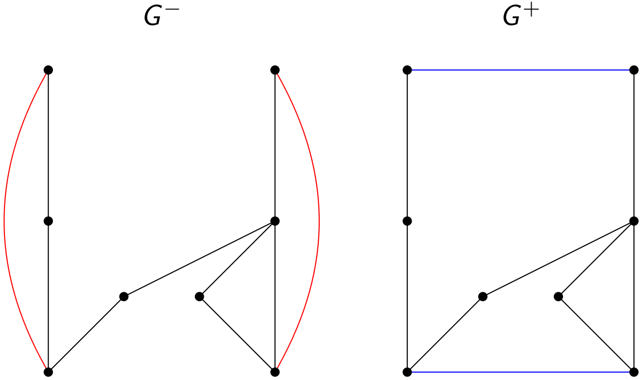
- **Major Difficulty:** Several key inequalities can fail

The point where old argument breaks down

Issue: the following key inequality is no longer true

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)}{\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)} \geq 1 + \epsilon'$$

The ratio is ≈ 0.82 in the following example:

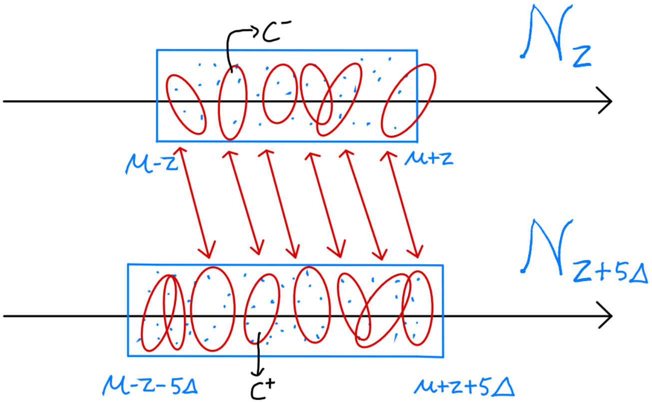


General case: refined switching idea

Definition: $\mathcal{N}_z = \{G \in \mathbf{d_n} : |X_{\mathrm{small}}(G) - \mu| \leq z\}$

Key Idea: Switching on clusters (=suitable sets of graphs)

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_z)}{\mathbb{P}(G^P_{\mathbf{d_n}} \in \mathcal{N}_{z+5\Delta})} \leq \frac{1}{1+\epsilon'}$$



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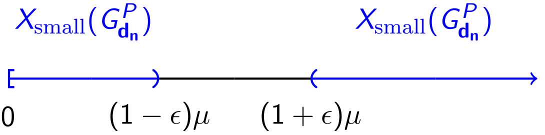
Conclusion: *whp number of small edges satisfies*



Summary

Degree-restricted random $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$

- Start with an empty graph on n vertices
- In each step: add one random edge to the graph,
  so that *the degree of each vertex v_i stays $\leq d_i$*

Main result: $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$ and uniform model $G_{\mathbf{d_n}}$ differ

If the bounded degree sequence $\mathbf{d_n}$ is not nearly regular, then can whp distinguish $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$ and random $\mathbf{d_n}$-graph $G_{\mathbf{d_n}}$

- *Proof technique: adapt switching method to stochastic process*

Open Question

Wormald's conjecture for 2-regular degree-restricted random process?