Large girth approximate Steiner Triple Systems

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Context of this talk

STS: Steiner Triple System (with n vertices)

Decomposition of complete n-vertex graph K_n into edge-disjoint triangles

- Existence question resolved by Kirkman (1847)
    - n-vertex STS exists $\Leftrightarrow$ $n \equiv 1, 3 \bmod 6$
- STS and generalizations (Designs) *testbed for new proof techniques*
    - e.g., Rödl (1982), Keevash (2014), Glock–Kühn–Lo–Osthus (2016)

This talk: Answer Erdős–Question (1973)

Show existence of approximate STS with arbitrary high 'girth'

Motivation

- High-Dimensional Combinatorics (Design/Hypergraph Theory)
- Interest/work of Lefmann–Phelps–Rödl (1993), Ellis–Linial (2013), Krivelevich–Kwan–Loh–Sudakov (2017)

Erdős–Questions on high-girth STS

Girth (for triple systems)

Smallest $g \geq 4$ for which there are g vertices spanning $\geq g-2$ triples

- Intuition: *large girth* $\cong$ *locally sparse*

Questions (Erdős, 1973)

(Q1) Are there STS with arbitrary high girth?
(Q2) Are there *partial STS* with arbitrary high girth and $\Theta(n^2)$ triples?

- *Partial STS* = edge-disjoint collection of triples (no decomposition)

Why is (Q2) interesting question?

- *Alteration Method* gives partial STS with $\geq c_\ell n^2$ triples and girth $> \ell$, with vanishing constant $c_\ell \to 0$ as $\ell \to \infty$ (Lefmann–Phelps–Rödl 1993)
- Any partial STS in which no 6 vertices span ≥ 3 triples has $o(n^2)$ triples (implied by $(6, 3)$-theorem of Ruzsa–Szemerédi 1978)

Main Result: High-girth approx. STS exist

We answer Erdős–Question (Q2) from 1973:

Bohman, W. (2018+): Approximate Steiner Triple Systems with high-girth

For every $\ell \geq 4$ there are $\beta, n_0 > 0$, such that, for all $n \geq n_0$
there are n-vertex partial STS with $\geq \left(1 - n^{-\beta}\right)\frac{n^2}{6}$ triples and girth $> \ell$

- Nearly best possible: n-vertex partial STS have $\leq \frac{1}{3}\binom{n}{2} \sim \frac{n^2}{6}$ triples
- *Algorithmic Proof:* analyze natural random greedy process
  (typically produces such high-girth partial STS)
- Result independently obtained by Glock–Kühn–Lo–Osthus

High-Girth Process

High-Girth Process ($\ell \geq 4$ is fixed)

Sequentially add random triples, chosen uniformly from all triples whose addition does not violate girth $> \ell$ constraint

- Terminates with a maximal partial STS
  (Girth $g > 4 \;\Leftrightarrow\;$ all triples are edge-disjoint)
- Enough to show that typically runs for $\geq \left(1 - n^{-\beta}\right)\frac{n^2}{6}$ steps

Equivalent description (of High-Girth Process)

Sequentially add random triples, chosen uniformly from all triples that
(i) are edge-disjoint from added triples, and
(ii) do not create 'forbidden' subhypergraph with $5 \leq v \leq \ell$ vertices

- Forbidden subhypergraph: v vertices that span $\geq v - 2$ triples
- Our analysis shows: constraint (i) has dominant effect

Glimpse of the Proof

High-Girth Process ($\ell \geq 4$ is fixed)

Sequentially add random triples, chosen uniformly from all triples that
(i) are *edge-disjoint from added triples*, and
(ii) do *not create 'forbidden' subhypergraph* with $5 \leq v \leq \ell$ vertices

- $\mathcal{H}_i$ = collection of triples added during first i steps
- $\mathcal{Q}_i$ = triples that can be added to $\mathcal{H}_i$ (without violating constraints)

Main Technical Result (Bohman, W., 2018+)

There is $\beta = \beta_\ell > 0$ such that, whp, for all steps $0 \leq i \leq (1 - n^{-\beta})\frac{n^2}{6}$

$$|\mathcal{Q}_i| \approx p^3 q \cdot \binom{n}{3} > 0$$

for suitable functions $p = p(t)$ and $q = q(t)$ that depend on time $t := i/n^2$

- $p^3 \approx \mathbb{P}$(triple f does not violate constraint (i) wrt. $\mathcal{H}_i$)
- $q \approx \mathbb{P}$(triple f does not violate constraint (ii) wrt. $\mathcal{H}_i$)

Pseudo-Random Heuristic

Heuristic Ansatz: $\mathcal{H}_i \approx$ random 3-uniform hypergraph

Setting time $t := i/n^2$, for all triples $f \in \binom{n}{3}$ we assume that

$$\mathbb{P}(f \in \mathcal{H}_i) \approx i/\binom{n}{3} \approx 6t/n$$

holds independently, unless constraint (i) or (ii) violated

- Functions $p(t)$ and $q(t)$ can be guessed by Pseudo-Random Heuristic

First example: $E_i = \binom{n}{2} \setminus \bigcup_{f \in \mathcal{H}_i} f$ = all edges 'untouched' by $\mathcal{H}_i$

Heuristic implies $\mathbb{P}(e \in E_i) \approx 1 - 6t =: p$

- $\mathbb{P}(f$ doesn't violate constraint (i) wrt. $\mathcal{H}_i) = \mathbb{P}(f \subseteq E_i) \approx p^3$

Second example: $\mathcal{Q}_i$ = 'available' triples (that can be added to $\mathcal{H}_i$)

Heuristic implies $\mathbb{P}(f \in \mathcal{Q}_i) \approx p^3 \cdot e^{-\sum_{5 \le j \le \ell} c_j t^{j-3}} =: p^3 \cdot q.$

Glimpse of the proof

Bohman, W. (2018+): Pseudo-Random Heuristic correct

There is $\beta = \beta_\ell > 0$ such that, whp, for all steps $0 \le i \le (1 - n^{-\beta})\frac{n^2}{6}$

$$|\mathcal{Q}_i| \approx p^3 q \cdot \binom{n}{3} > 0$$

where, using rescaled time $t = i/n^2$,

$$p = p(t) := 1 - 6t \qquad \text{and} \qquad q = q(t) := \prod_{5 \le j \le \ell} e^{-c_j t^{j-3}}$$

Surprising feature

- $p \to 0$ and $q > 0$ as $t = i/n^2 \to 1/6$
  (heuristically explains why partial STS constraint (i) dominates)

Proof based on Differential-Equation-Method

- requires tracking more variables
  ('routes' to forbidden subhypergraphs)

SUMMARY

Approximate STS with arbitrary high girth

For every fixed $\ell \geq 4$ and n large enough, there are n-vertex partial Steiner Triple Systems with $(1 - o(1))n^2/6$ triples and girth $> \ell$

Remarks

- *Answers Erdős–Question from 1973*
- Algorithmic Proof: via natural random greedy process (iteratively add random triples that keep girth $> \ell$)

Questions

- Does greedy algorithm terminate with $n^2/6 - n^{3/2+o(1)}$ triples?
- What can we say about growing girth $\ell = \ell(n) \to \infty$?