Phase transition in evolving random graphs with bounded degrees

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Abstract

We study the phase transition of the random d-process, which is a time-evolving random graph model with bounded degrees: starting with an empty graph on n vertices, new random edges are added step-by-step so that the maximum degree remains at most d. We show that the d-process undergoes a giant component phase transition for fixed $d \geqslant 3$, solving a problem of Wormald from 1999. We also show that the giant component has a non-trivial distribution at the end of the 2-process, verifying a conjecture of Balińska and Quintas from 1993. Our techniques show that the critical point of the phase transition in fact equals the blow-up point of the susceptibility, which in turn is determined by $O(d^2)$ many differential equations. Our results extend to many hypergraph generalizations of the d-process.

:intro

1 Introduction

One of the most interesting features of random graphs is the 'giant component' phase transition, i.e., the sudden change of the global graph structure from only small components to a single giant component plus small ones. Motivated by properties of real-world networks and ideas from mathematical physics, during the last two decades there has been an increased interest in the phase transition of time-evolving random graph models. For many of these difficult-to-analyze models, it remains a mathematical challenge to narrow the widening gap between simulation based results and theoretical understanding.

In this paper we study a time-evolving random graph model with bounded degrees. More precisely, for an integer parameter $d \geqslant 1$ we consider the *random d-process* $(G_i)_{i\geqslant 0} = (G^d_{n,i})_{i\geqslant 0}$ with vertex set $[n] = \{1, \dots, n\}$, that starts with no edges and then sequentially adds new edges one-by-one, each time choosing the next edge uniformly at random from all so-far unused edges whose addition does not create a vertex of degree d; this process stops when no more such edges can be added, which whp[1] happens after $\lfloor dn/2 \rfloor$ steps, see [42]. Note that $G^d_{n,i}$ has i edges. Furthermore, for $d \geqslant n-1$ the d-process reduces to the classical Erdős-Rényi random graph process (which simply adds a new random edge in each step).

The random d-process is so natural that it has been studied by different communities. In combinatorics it can be traced back to a suggestion[2] of Erdős and Rényi from 1961 for more realistic modeling [22]. In chemistry it has been studied as early as 1985, motivated by polymerization modeling [33, 7]. It also corresponds to a simple random greedy algorithm for generating d-regular graphs [42, 52] (that differs from the usual uniform random d-regular graph model). More recently, it has also been analyzed through the lens of statistical physics [10], and a square lattice variant was studied in percolation theory [20]. Further interest in the d-process stems from the fact that its analysis has repeatedly stimulated the development of new proof techniques, including the differential equation method [42, 54, 53] and associated self-correction idea [48], which both were instrumental for later breakthroughs in Ramsey Theory [12, 13, 23].

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[1] As usual, we say that an event holds *whp* (with high probability) if it holds with probability tending to 1 as $n \to \infty$.

[2] Erdős and Rényi proposed, more generally, to study random graph processes where the probability of joining two vertices v and w in the next step should depend on the current degrees of these vertices, see [22, p. 344].

Establishing the phase transition in the random d-process is an open problem of Wormald from 1999: he asked [53, Section 4.3] whether there is a critical point $t_c = t_c(d)$ after which a linear size 'giant' component emerges, i.e., such that the size $L_1(i) = L_1(G_i)$ of the largest component after i steps is whp $L_1(\lfloor tn \rfloor) = o(n)$ when $t < t_c$ and whp $L_1(\lfloor tn \rfloor) = \Theta(n)$ when $t > t_c$. Turning to the history of this problem, results of Ruciński and Wormald [44] from 2002 imply that if t_c exists, then $t_c \leqslant (d/2 - 1/6)$ for $d \geqslant 3$. In 2008 Droseltis [21] attempted to analyze the d-process phase transition using the configuration model, and in 2011 Ben-Naim and Krapivsky [10] presented kinetic theory based evidence for the existence of t_c for $d \geqslant 3$. In 2013 Seierstad [45] showed that if t_c exists (and further technical properties hold), then the size of the giant component is asymptotically normal after suitable rescaling. Around 1993 Balińska and Quintas [8, 9] also made a simulation based conjecture regarding the distribution of the giant component in the 2-process.

In this paper we solve the more than 20-year-old problem of Wormald, by establishing that there is a giant component phase transition in the d-process for fixed $d \geqslant 3$ (see Theorem 1.1). We also verify the more than 25-year-old conjecture of Balińska and Quintas, by relating the size of the giant component at the end of the 2-process with the largest component in random 2-regular graphs (see Theorem 1.3). In addition, we show that the d-process with $d = d(n) \to \infty$ as $n \to \infty$ has the same critical point $t_c = 1/2$ as the Erdős–Rényi random graph process (see Remark 1.2), and establish a phase transition in many hypergraph generalizations of the d-process, including variants where each vertex has its own degree bound (see Theorems 1.5–1.6).

The motivation for this work is to further develop the emerging theory of phase transition in time-evolving random graph models. Powerful ideas and heuristics from percolation theory [1, 36] and aggregation and coagulation theory [3, 34] suggest the following generic two-step program [2, 4, 46] for establishing phase transition: (i) show that the so-called susceptibility has a blow-up point t_c, and (ii) show that this t_c coincides with the critical point after which a giant component emerges; here the *susceptibility* is defined as the expected size of the component containing a randomly chosen vertex, i.e.,

$$S(i) = S(G_i) := \sum_{v \in [n]} |C_v(G_i)|/n, \tag{1.1}$$ def:S

where $C_v(G_i)$ denotes the component of G_i which contains vertex v; cf. [30, 28]. While (parts of) this emerging program have been implemented for a few random graph models [2, 4, 14, 47, 32, 41], its application often remains challenging. For example, step (i) usually requires showing that some system of differential equations has a blow-up point t_c, which for the d-process is non-trivial since this involves $O(d^2)$ many equations. Our proofs show how to avoid the differential equation analysis in the two-step phase transition program, i.e., how to obtain useful information about the susceptibility blow-up point t_c without much technical effort.

1.1 Main results: random d-process

Our first result proves that the random d-process undergoes a phase transition for fixed $d \geqslant 3$, i.e., that there is a critical point $t_c = t_c(d)$ at which the size of the largest component changes from order $O(\log n)$ to order $\Theta(n)$; this solves a problem of Wormald from 1999. Recall that the d-process whp ends after $\lfloor dn/2 \rfloor$ steps.

:maind **Theorem 1.1** (Phase transition of d-process for $d \geqslant 3$)**.** *Given $d \geqslant 3$, there exists $t_c \in (0, d/2)$ such that, in the random d-process, for any $t \in [0, d/2]$ the size of the largest component after $\lfloor tn \rfloor$ steps whp satisfies*

$$L_1\big(\lfloor tn \rfloor\big) = \begin{cases} O(\log n) & \text{if } t < t_c, \\ \Theta(n) & \text{if } t > t_c. \end{cases} \tag{1.2}$$ eq:maind:L1

There also exists a continuously increasing function $s : [0, t_c) \to [1, \infty)$ with $\lim_{t \nearrow t_c} s(t) = \infty$ such that, in the random d-process, for any $t \in [0, t_c)$ the susceptibility after $\lfloor tn \rfloor$ steps whp satisfies $S(\lfloor tn \rfloor) \sim s(t)$.

ind:tc **Remark 1.2.** *The critical point of the random d-process satisfies $t_c = t_c(d) \to 1/2$ as $d \to \infty$. In fact, for any $t \in [0, \infty)$ whp (1.2) also holds with $t_c = 1/2$ when $d = d(n) \to \infty$ as $n \to \infty$. (See Appendix A.)*

This result recovers several fundamental phase transition features of the classical Erdős-Rényi reference model [16, 31, 30] where (a) the size of the largest component after $\lfloor tn \rfloor$ steps also whp satisfies (1.2) with $t_c = 1/2$, and (b) the susceptibility after $\lfloor tn \rfloor$ steps is whp asymptotic to $1/(1 - 2t)$, i.e., also blows up

at t_c. Remark 1.2 indicates that the phase transition of the d-process approaches Erdős-Rényi behavior for large d (as expected, since then the degree bounds ought to become negligible), illustrating that small d is the most interesting case. In fact, the assumption $d \geqslant 3$ in Theorem 1.1 is best possible, as we shall see.

From the perspective of mathematical physics, Theorem 1.1 shows that in the d-process the following two different phase transition thresholds are in fact the same: (i) the critical point around which the giant component emerges, and (ii) the blow-up point of the susceptibility. This mimics deep results in percolation theory [1, 36], which show that two different critical probabilities related to (i) and (ii) are the same. This also mimics heuristics in aggregation and coagulation theory [3, 34], which suggest that two different definitions of the gelation time related to (i) and (ii) are the same. Furthermore, since the idealized susceptibility $s(t)$ is in fact determined by a system of $O(d^2)$ many differential equations (see Remark 1.7), we can use equality of (i) and (ii) to estimate the critical point t_c by numerically solving this system; see Section 5 for the details.

Our second result proves that the random 2-process does not exhibit Erdős-Rényi like behavior. Indeed, Theorem 1.3 shows that the rescaled size of the largest component $L_1(n)/n$ converges in distribution to a non-trivial random variable; this in particular verifies the conjecture $\mathbb{P}(L_1(n) > n/2) \sim \log(\sqrt{2}+1) \approx 0.8814$ of Balińska and Quintas from 1993. Recall that the 2-process whp ends after n steps.

thm:d2 **Theorem 1.3** (Largest component in final graph of 2-process)**.** *There exists a continuously decreasing function $F : (0,1] \to [0,1]$ such that, in the random 2-process, the size of the largest component after n steps satisfies*

$$\lim_{n\to\infty} \mathbb{P}\big(L_1(n)/n > c\big) \;=\; F(c) \qquad \text{for any } c \in (0,1], \tag{1.3}$$

eq:L1sup:d2

with $F(c) \in (0,1)$ for $c \in (0,1)$, $F(c) = \log\big(\sqrt{c^{-1}} + \sqrt{c^{-1}-1}\big)$ for $c \in (1/2,1]$, and $F(\varepsilon) \to 1$ as $\varepsilon \to 0$.

rem:d2 **Remark 1.4.** *Analogous to (1.2), for any $t \in [0,1)$ whp $L_1(\lfloor tn \rfloor) = O(\log n)$ in the random 2-process.*

It is tantalizing that the probability (1.3) in the 2-process has the same limit as the corresponding probability $\lim_{n\to\infty} \mathbb{P}(L_1(R_n)/n > c) = F(c)$ in the standard configuration model R_n for random 2-regular graphs with n vertices [15, 52], see Lemma 4.3. Our proof of Theorem 1.3 explains this by establishing a close connection between R_m and the way paths merge during the final $m = o(n)$ steps of the 2-process, see Section 4. The complementary distribution function F also arises in the context of the Ewens's sampling formula, and is closely related to the Poisson-Dirichlet distribution with parameter $1/2$; see [5, Sections 5.5 and 6.1].

Our phase transition results extend to the hypergraph generalizations of the d-process introduced in [25]. More precisely, the *random k-uniform d-process* process with vertex set $[n]$ sequentially adds new k-uniform hyperedges one-by-one (starting with no hyperedges), such that the next hyperedge is chosen chosen uniformly at random from all so-far unused hyperedges whose addition does not create a vertex[3] of degree d; this process stops when no more such edges can be added, which whp happens after $\lfloor dn/k \rfloor$ steps, see [25].

maindk **Theorem 1.5** (Phase transition of k-uniform d-process)**.** *Given $k, d \geqslant 2$ with $(k,d) \neq (2,2)$, Theorem 1.1 also holds (with $d/2$ replaced by d/k) for the random k-uniform d-process.*

This result implies that, among k-uniform d-processes, only the trivial k-uniform 1-process (which simply adds disjoint edges) and the 2-process exhibit special phase transition behaviors.

1.2 Organization

The remainder of this paper is organized as follows. In Section 1.3 we give a detailed overview of the proof strategy, highlighting the key ingredients and steps. In Section 1.4 we then state our main phase transition result Theorem 1.6, which applies to many degree restricted hypergraph generalizations of the d-process (and extends Theorems 1.1 and 1.5). The next two sections are the core of the paper: in Section 2 we use coupling arguments to obtain some approximate control over the degree restricted process, and in Section 3 we then combine this with the differential equation method in order to prove Theorem 1.6, by relating the evolution of the degree restricted process with suitable differential equations. In Section 4 we consider the 2-process: after a detailed outline of the proof strategy, we then prove Theorem 1.3 by using coupling arguments to relate the evolution of the 2-process with the 2-regular configuration model. In Section 5 we demonstrate how to estimate the value of the critical point t_c by numerically solving relevant differential equations. In

[3] For a hypergraph H, the degree of vertex v is simply defined as the number of edges of H that contain v.

Section 6 we discuss several open problems. Finally, Appendix A studies the critical point $t_c = t_c(d)$ of the d-process for large d, Appendix B shows how to deduce results for the degree restricted process from results for its multigraph variant, and Appendix C contains some proofs that are omitted from the main text.

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1.3 Proof overview

We now outline our proof of Theorem 1.1, which adapts the mathematical physics inspired two-step phase transition program to the random d-process $(G_i)_{i\geqslant 0} = (G^d_{n,i})_{i\geqslant 0}$ with $d \geqslant 3$. Recalling that t_c will denote the blow-up point of the susceptibility (whose existence will be established as part of the proof), the basic strategy is to first carefully analyze the first $(t_c - \varepsilon)n$ steps, say, and then use rough approximation arguments to show that a giant component emerges by step $(t_c + \varepsilon)n$.

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1.3.1 Key ingredients

We shall distinguish two types[4] of vertices: *active* vertices with degree less than d and *inactive* vertices with degree equal to d, the crux being that the d-process only adds new edges between active vertices. In addition, our core two-step argument will rely on the following three key ingredients:

• **Susceptibility variables:** Taking into account that inactive vertices play no role in the subsequent evolution of the d-process, instead of the usual susceptibility we here study the *active susceptibility*

$$S_u(i) = S_u(G_i) := \sum_{v\in[n]} |C_v(G_i) \cap \mathcal{U}_i|/|\mathcal{U}_i|, \tag{1.4}$$

def:Su

where $\mathcal{U}_i$ denotes the active vertex-set of G_i, i.e., the set of vertices with degree less than d. In words, $S_u(i)$ denotes the expected active size of the component containing a randomly chosen active vertex from G_i. Using the differential equation method [54, 53, 51], our aim is to show that for $t < t_c$ whp

$$S_u(tn) \approx s_u(t), \tag{1.5}$$

eq:Su:approx

where the function $s_u(t)$ is the solution to suitable differential equations, which arise from considering the expected one-step changes $\mathbb{E}(S_u(i+1) - S_u(i) \mid G_i)$. This is more involved than in previous susceptibility related work [2, 4, 14, 47, 32, 11]: to even be able to write down these expected changes we need to introduce additional $O(d^2)$ auxiliary random variables $Z_{x,y}(i)$, which are refinements of $S_u(i)$ that take into account the vertex degrees inside each component (intuitively, degree knowledge is required for understanding when active vertices can become inactive); see (3.2) for the precise definitions of $Z_{x,y}(i) = Z_{x,d,y,d}(i)$.

• **Blow-up point t_c:** For the active susceptibility we obtain differential equations of form

$$s'_u(t) = F\Big(t,\ s_u(t),\ \big(z_{x,y}(t)\big)_{0\leqslant x,y<d}\Big), \tag{1.6}$$

eq:su:diff

along with $O(d^2)$ differential equations for the functions $z_{x,y}(t) = z_{x,d,y,z}(t)$ associated with the new variables $Z_{x,y}(i)$; see (3.11) and (3.13)–(3.17) for the actual equations. This system of differential equations is larger and more complicated than in previous susceptibility related work [4, 14, 47, 32, 11], where the blow-up point of such systems is established by technical analysis. We manage to sidestep this difficulty by adapting ideas developed for processes where differential equations are unavailable [41], so that we can show existence of the critical point t_c where $s_u(t)$ blows up without much technical effort (by transferring properties of the random d-process to the differential equations), see (1.10)–(1.11) and Section 3.1 for more details.

• **Rough control:** Using coupling arguments, our aim is to show that (for a small number of steps) the evolution of the d-process can be sandwiched between two simpler random graph models. Exploiting combinatorial properties of the d-process, for sufficiently small m we essentially show that whp

$$E(G_i) \cup \mathcal{M}_{m/2,\mathcal{U}_i} \ \subseteq\ E(G_{i+m}) \ \subseteq\ E(G_i) \cup \mathcal{E}_{2m,\mathcal{U}_i}, \tag{1.7}$$

eq:sandwich

where $\mathcal{E}_{\ell,\mathcal{U}_i}$ and $\mathcal{M}_{\ell,\mathcal{U}_i}$ intuitively denote ℓ random (matching) edges with endvertices in the active vertex set $\mathcal{U}_i$, see Lemma 2.2. The usage of matching edges in these inclusions differs from previous related coupling

[4] These two types are called unsaturated and saturated vertices in [42, 43, 52, 44].

arguments [14, 47, 32, 41]. The crux is that via branching process and random walk arguments we can obtain some rough control over the two random graphs models on both sides of (1.7), see Lemma 2.3. In particular, assuming that the graph G_i is well-behaved, for small m this essentially allows us to conclude that whp

$$L_1(i+m) = \begin{cases} O(\log n) & \text{if } m \leqslant c|\mathcal{U}_i|/S_u(i), \\ \Theta(n) & \text{if } m \geqslant C|\mathcal{U}_i|/S_u(i), \end{cases} \tag{1.8}$$

eq:sandwich:L1

where $c, C > 0$ are suitable constants; see Theorem 2.1 for the full details.

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1.3.2 Phase transition: two-step approach

With these key ingredients in hand, we then are in a position to prove the phase transition result Theorem 1.1 for the d-process with $d \geqslant 3$ using (a rigorous version of) the following two-step approach:

Step 1: Subcritical phase. The goal is to establish both (a) the existence of the active susceptibility blow-up point t_c and (b) that the largest component whp has size $O(\log n)$ after $(t_c - \varepsilon)n$ steps. To this end we shall use a 'nibble' argument from probabilistic combinatorics that develops ideas from [47, 41], i.e., show that certain structural properties hold by iteratively considering a small number of steps of the d-process (where the number of steps considered decreases as the active susceptibility increases). More precisely, using the differential equation method and the rough control bounds we first use induction on $h \geqslant 0$ to essentially establish that, after $i_h \approx t_h n$ steps, the resulting graph G_{i_h} is whp well-behaved in the sense that

$$S_u(i_h) \approx s_u(t_h), \quad |\mathcal{U}_{i_h}| \approx u(t_h)n, \quad \text{and} \quad L_1(i_h) = O(\log n), \tag{1.9}$$

eq:dem:temp

where the deterministic time-sequence $(t_h)_{h \geqslant 0}$ satisfying $t_h < d/2$ is roughly of form

$$t_0 = 0 \quad \text{and} \quad t_{h+1} \approx t_h + \xi u(t_h)/s_u(t_h), \tag{1.10}$$

eq:th:temp

see Theorem 3.1 for the full details (where we in fact inductively establish that the component distribution of G_{i_h} has exponentially decaying tails). To motivate the form of the t_h, note that by the approximations from (1.9) we have $i_{h+1} - i_h < c|\mathcal{U}_{i_h}|/S_u(i_h)$ for $\xi = \xi(c)$ small enough, which eventually enables us to inductively establish $L_1(i_{h+1}) = O(\log n)$ by invoking the rough control bound (1.8) with $i = i_h$ and $m = i_{h+1} - i_h$, see Section 3.4. More importantly, the form (1.10) of the $t_h < d/2$ automatically guarantees that

$$\lim_{h \to \infty} u(t_h)/s_u(t_h) = 0, \tag{1.11}$$

eq:blowup:temp

and it also is not hard to see that $\lim_{h\to\infty} s_u(t_h) = \infty$, i.e., that the idealized active susceptibility blow-ups up at the critical point $t_c := \lim_{h\to\infty} t_h \in (0, d/2]$, see Sections 3.1 and 3.2.3 for the details (where we also show that the usual susceptibility blows up at t_c). We then pick $h = h(\varepsilon, t_c)$ large enough such that $t_h > t_c - \varepsilon$, which by our inductive bound (1.9) suggests that whp

$$L_1\big((t_c - \varepsilon)n\big) \leqslant L_1\big(i_h\big) = O(\log n), \tag{1.12}$$

eq:choiceh:L1bound

as claimed by the $t < t_c$ part of (1.2) from Theorem 1.1; see Section 3.2.1 for the rigorous details.

Step 2: Supercritical phase. The goal is to establish that the largest component whp has size $\Theta(n)$ after $(t_c + \varepsilon)n$ steps. To this end we shall use a 'sprinkling' argument from random graph theory, i.e., show emergence of the desired giant component by adding εn extra edges to a suitable subcritical graph G_{i_h}. More precisely, we first pick $h = h(\varepsilon, C)$ large enough such that the approximations (1.9) and the limit (1.11) together guarantee that the graph G_{i_h} after $i_h \approx t_h n$ steps is whp well-behaved and satisfies

$$C|\mathcal{U}_{i_h}|/S_u(i_h) \approx Cu(t_h)n/s_u(t_h) < \varepsilon n. \tag{1.13}$$

eq:choiceh:temp

Invoking the rough control bound (1.8) with $i = i_h$ and $m \approx \varepsilon n$, using $t_h \leqslant t_c$ this suggests that whp

$$L_1\big((t_c + \varepsilon)n\big) \geqslant L_1(i_h + \varepsilon n) = \Omega(n), \tag{1.14}$$

eq:choiceh:L1bound

as claimed by the $t > t_c$ part of (1.2) from Theorem 1.1; see Section 3.2.2 for the rigorous details. It is instructive that (1.14) fails for the random 2-process, which already ends after at most $dn/2 = n = t_c n$ steps (a similar remark applies to the 1-process). For the d-process with $d \geqslant 3$, this explains why we also need to show $t_c < d/2$ via an extra argument, see Section 3.2.3. For the 2-process this 'end of process' obstacle also explains why Theorem 1.3 requires a different two-step proof approach, whose details we defer to Section 4.

1.4 More general results: random k-uniform $\mathbf{d}_n$-process

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Our arguments extend to a large class of degree restricted hypergraph processes that generalize the d-process. Informally, each vertex $v \in [n]$ has its own maximum degree restriction $d_v^{(n)} \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and in each step we add one new random k-hyperedge that does not violate any of these degree restrictions. More formally, given $k \geqslant 2$ and a degree sequence $\mathbf{d}_n = \big(d_1^{(n)}, \ldots, d_n^{(n)}\big) \in \mathbb{N}^n$, the *random k-uniform $\mathbf{d}_n$-process* corresponds to a random sequence $(H_i)_{i\geqslant 0} = (H_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$ of k-uniform hypergraphs with vertex set $[n]$ and empty initial initial hypergraph H_0. In each step we obtain obtain H_{i+1} by adding the hyperedge e_{i+1} to H_i, where e_{i+1} is chosen uniformly at random from all hyperedges in $\binom{\mathcal{U}_i}{k} \setminus E(H_i)$, where the active vertex set $\mathcal{U}_i$ contains all vertices $v \in [n]$ with degree less than $d_v^{(n)}$ in H_i; this process stops when no more such edges can be added, which happens[5] after $\mathbf{m}_n - O(1)$ steps, where $\mathbf{m}_n := \sum_{v\in[n]} d_v^{(n)}/k$.

Our main result regarding the phase transition of the random k-uniform $\mathbf{d}_n$-process extends Theorems 1.1 and 1.5, based on two main assumptions: (a) that all degree restrictions in $\mathbf{d}_n$ are bounded by some constant Δ, and (b) that the fraction of vertices in $\mathbf{d}_n$ with degree restriction j converges to r_j. For example, the d-process corresponds to the special case $k = 2$, $\Delta = d$, $\mathbf{d}_n = (d, \ldots, d)$ and $r_j = \mathbb{1}_{\{j=d\}}$, where $T = d/2$. In view of (1.15), note that the k-uniform $\mathbf{d}_n$-process ends after $\mathbf{m}_n - O(1) = Tn - o(n)$ steps, where $T > 0$ follows from $\sum_{j\in[\Delta]} r_j > 0$. Here $N_{\geqslant j}(H)$ denotes the number of vertices of H in components of size at least j.

mainhg

Theorem 1.6 (Phase transition of k-uniform $\mathbf{d}_n$-process)**.** *Given $k \geqslant 2$, $\Delta \geqslant 1$ and $\mathbf{r} = (r_0, \ldots, r_\Delta) \in [0,1]^{\Delta+1}$ with $\sum_{j\in[\Delta]} r_j > 0$, set $T := \sum_{j\in[\Delta]} j r_j/k$. Then there exist $t_c \in \big(0, \min\{T, \frac{1}{k-1}\}\big]$ and a continuously increasing function $s : [0, t_c) \to [1, \infty)$ such that the following holds for the random k-uniform $\mathbf{d}_n$-process provided that $\mathbf{d}_n = \big(d_1^{(n)}, \ldots, d_n^{(n)}\big)$ satisfies $\mathbf{d}_n \in \{0, \ldots, \Delta\}^n$ and*

$$\lim_{n\to\infty} \big|\big\{v \in [n] : d_v^{(n)} = j\big\}\big|/n \;=\; r_j \qquad \text{for each } 0 \leqslant j \leqslant \Delta. \tag{1.15}$$

conv

:subcr

1. (Subcritical phase: exponential tails) *For every $t \in [0, t_c)$ there are constants $a, A > 0$ depending only on $t, k, \Delta, \mathbf{r}$ such that, with probability $1 - o(n^{-99})$, we have $N_{\geqslant j}(H_{n,\lfloor tn\rfloor}^{k,\mathbf{d}_n}) \leqslant Ae^{-aj}n$ for all $j \geqslant 1$, which in particular implies $L_1(H_{n,\lfloor tn\rfloor}^{k,\mathbf{d}_n}) \leqslant C \log n$ for a suitable constant $C > 0$.*

:supcr

2. (Supercritical phase: giant component) *For every $t \in (t_c, T)$ there is a constant $c = c(t, k, \Delta, \mathbf{r}) > 0$ such that, with probability $1 - o(n^{-99})$, we have $L_1(H_{n,\lfloor tn\rfloor}^{k,\mathbf{d}_n}) \geqslant cn$.*

nhg:tc

3. (Susceptibility: blow-up at critical point t_c) *For every $t \in [0, t_c)$ there is a parameter $\xi_n > 0$ with $\xi_n = o(1)$ such that, with probability $1 - o(n^{-99})$, we have $S(H_{n,i}^{k,\mathbf{d}_n}) = (1 \pm \xi_n)\, s(i/n)$ for all $0 \leqslant i \leqslant tn$. Furthermore, $\lim_{t\nearrow t_c} s(t) = \infty$ if $t_c < T$.*

nhg:tc

Remark 1.7. *The function $s(t)$ is determined by the unique solutions to a system of $O(\Delta^4)$ differential equations, see* (3.11) *and* (3.13)–(3.17).

This result establishes exponential decay of the component size distribution in the subcritical case $t \in [0, t_c)$, which is well-known for the Erdős-Rényi reference model [16, 31]. This also strengthens analogies with Percolation theory [26, 19], where exponential decay is a prominent hallmark of the subcritical phase.

Theorem 1.6 identifies the critical point t_c of the phase transition as the blow-up point of the susceptibility, but it does so only when $t_c < T$, i.e., only if the giant component emerges when there is still some significant part of the degree restricted process remaining. The 1-process shows that this extra complication is necessary, since its susceptibility never blows up (as all components trivially have size at most 2). Theorem 1.6 also implies that the separation $t_c < T$ follows from the sufficient condition $T > 1/(k-1)$, which in turn is equivalent to the idealized average degree condition $\sum_{j\in[\Delta]} j r_j > 1 + 1/(k-1)$. This readily implies $t_c < T$ for the d-process with $d \geqslant 3$ and for the k-uniform d-process with $(k, d) \neq (2, 2)$ and $k, d \geqslant 2$, which in turn demonstrates that Theorem 1.6 implies Theorems 1.1 and 1.5 (recalling that these two processes whp end after $\lfloor dn/2\rfloor$ and $\lfloor dn/k\rfloor$ steps, respectively). In fact, the aforementioned sufficient condition for $t_c < T$ is sharp due to the 2-process, which satisfies $T = 1/(k-1) = t_c$ and $\sum_{j\in[\Delta]} j r_j = 1 + 1/(k-1)$.

[5] The process stops when we cannot add another any edge without violating the degree restriction of some vertex v. Then there must be $|\{v \in [n] : d_v^{(n)} = j\}| - O(1)$ vertices with degree $j \in [\Delta]$, so the process stops after adding $\mathbf{m} - O(1)$ edges.

1.4.1 Multigraph variant

igraph

In the proofs it is convenient to consider a *multigraph variant of the k-uniform $\mathbf{d}_n$-process*, which corresponds to a random sequence $(G_i)_{i\geqslant 0} = (G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$ of hypergraphs with vertex set $[n]$ and empty initial hypergraph G_0. In each step we obtain G_{i+1} by adding the hyperedge $e_{i+1} = \{v_{i+1,1}, \dots, v_{i+1,k}\}$ to G_i, where the vertices $v_{i+1,j}$ are independently chosen uniformly at random from the active vertex-set $\mathcal{U}_i$ of G_i; this process stops when no more such edges can be added. While these hypergraphs may contain multiple edges or edges with repeated vertices, there will be rather few of these. As usual, it thus suffices to prove Theorem 1.6 for the multigraph variant, since this result then transfers in a routine way to the normal k-uniform $\mathbf{d}_n$-process; we include the fairly standard details of this reduction in Appendix B.

:multi

Remark 1.8. *It suffices to prove Theorem 1.6 for the multigraph variant $(G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$.*

2 Couplings: approximating the evolution

upling

In this preparatory section we consider the multigraph variant $(G_i)_{i\geqslant 0} = (G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$ of the k-uniform $\mathbf{d}_n$-process defined in Section 1.4.1, and make the heuristic 'rough control' bound (1.8) bound rigorous. In particular, Theorem 2.1 gives, starting from the hypergraph G_i after i steps, some approximate control over the components arising in the subsequent evolution (during a small number of steps). Defining $N_j(G_i)$ as the number of vertices of G_i in components of size j, our main technical assumption is that G_i is well-behaved in the sense that its component distribution has exponential tails: $\sum_{j\in[n]} \beta^j N_j(G_i) \leqslant Bn$ implies $N_{\geqslant j}(G_i) \leqslant B\beta^{-j}n$, which also gives $L_1(G_i) = O(\log n)$. The subcritical statement (2.1) is optimized for our upcoming inductive proof of Theorem 1.6, implying that whp all subgraphs of G_{i+m} remain well-behaved if $m \leqslant c\cdot|\mathcal{U}_i|/S_u(i)$ for small $c > 0$. The supercritical statement (2.2) implies that G_{i+m} whp contains a giant component if $m \geqslant C\cdot|\mathcal{U}_i|/(S_u(i)-1)$ for large $C > 0$. Note that $N_j(G_i)$ and $|\mathcal{U}_i|$ are determined by G_i.

:rough

Theorem 2.1 (Rough control)**.** *Given $k \geqslant 2$, $\beta > 1$ and $B, \xi, \pi > 0$, there are $\alpha > 1$ and $A, c, \sigma, n_0 > 0$ such that the following holds for all $n \geqslant n_0$ and $\mathbf{d}_n \in \mathbb{N}^n$, writing $(G_i)_{i\geqslant 0} = (G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$. If $\sum_{j\in[n]} \beta^j N_j(G_i) \leqslant Bn$ and $\xi n \leqslant m \leqslant |\mathcal{U}_i|/8k^2$, then*

$$\mathbb{P}\big(\textstyle\sum_{j\in[n]} \alpha^j N_{\geq j}(G_{i+m}) \leqslant An \;\big|\; G_i\big) \geqslant 1 - n^{-\pi} \qquad \text{if } k(k-1)\cdot m/|\mathcal{U}_i|\cdot S_u(i) \leqslant 1/4, \tag{2.1}$$

thm:rough:sub

$$\mathbb{P}\big(L_1(G_{i+m}) \geqslant cn \;\big|\; G_i\big) \geqslant 1 - e^{-\sigma n} \qquad \text{if } k(k-1)\cdot m/|\mathcal{U}_i|\cdot (S_u(i)-1) \geqslant 4. \tag{2.2}$$

thm:rough:super

The constants $1/4$ and 4 in (2.1)–(2.2) suffice for our purposes, and we remark that they could be improved to $1-O(\varepsilon)$ and $1+O(\varepsilon)$ under the stronger assumption $m = \Theta(\varepsilon|\mathcal{U}_i|)$. The ad-hoc restriction $m \geqslant \xi n$ naturally arises in our later applications, but could also be weakened. The proof of Theorem 2.1 combines coupling arguments with branching process and random walk arguments, and is spread across Sections 2.1–2.3. On a first reading, the reader may perhaps wish to skip to the main phase transition proof in Section 3.

2.1 Sandwiching between two simpler models

ndwich

The key step in the proof of Theorem 2.1 is to show that, starting from the hypergraph G_i after i steps, we can whp sandwich the subsequent evolution of the multigraph process between two simpler random hypergraph models. To this end we define $\mathcal{E}_{\ell,W}$ as the set $\{f_1, \dots, f_\ell\}$ of edges with $f_j = \{w_{j,1}, \dots, w_{j,k}\}$, where each vertex $w_{j,h} \in W$ is chosen independently and uniformly at random. Similarly, we write $\mathcal{M}_{\ell,W}$ for a uniformly chosen random k-matching of W of size $|\mathcal{M}_{\ell,W}| = \ell$. Noting that $\mathcal{U}_i$ is determined by G_i, the point of the inclusions in (2.3)–(2.4) is that they enable us to study monotone properties of G_{i+j} via two more tractable models (see Section 2.2); below we use the shorthand $G + \mathcal{E} := (V(G),\, E(G)\cup\mathcal{E})$.

ndwich

Lemma 2.2. *Given $k \geqslant 2$, there is $\lambda > 0$ such that the following holds for all $n \geqslant 1$ and $\mathbf{d}_n \in \mathbb{N}^n$, writing $(G_i)_{i\geqslant 0} = (G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$. If $0 \leqslant m \leqslant |\mathcal{U}_i|/(4k^2)$, then there is a coupling of $G_{i+\lceil m/2\rceil}$ and $\mathcal{E}_{m,\mathcal{U}_i}$ and a coupling of G_{i+M} and $\mathcal{M}_{\lceil m/2\rceil,\mathcal{U}_i}$ such that*

$$\mathbb{P}\big(G_{i+\lceil m/2\rceil} \subseteq G_i + \mathcal{E}_{m,\mathcal{U}_i} \;\big|\; G_i\big) \geqslant 1 - e^{-\lambda m}, \tag{2.3}$$

eq:sandwich:super

$$\mathbb{P}\big(G_i + \mathcal{M}_{\lceil m/2\rceil,\mathcal{U}_i} \subseteq G_{i+m} \;\big|\; G_i\big) \geqslant 1 - e^{-\lambda m}. \tag{2.4}$$

eq:sandwich:sub

Proof. Inequalities (2.3)–(2.4) are trivial for $m = 0$, so we henceforth assume $m \geqslant 1$. We consider $(f_j)_{1\leqslant j\leqslant m}$ with $f_j = \{w_{j,1}, \dots, w_{j,k}\}$, where each vertex $w_{j,h} \in \mathcal{U}_i$ is chosen independently and uniformly at random. Clearly, $\{f_1, \dots, f_m\}$ gives $\mathcal{E}_{m,\mathcal{U}_i}$. Furthermore, starting from G_i we obtain the subsequent evolution of the degree restricted multigraph process by sequentially traversing the $(f_j)_{1\leqslant j\leqslant m}$, only adding those edges which do not violate the degree restrictions of any vertex. The crux is that any active vertex in $\mathcal{U}_i$ can be adjacent to at least one more edge, which implies that all edges in any induced k-matching $\mathcal{M} \subseteq \{f_1, \dots, f_m\} = \mathcal{E}_{m,\mathcal{U}_i}$ are added by the degree restricted process (the vertices of $f \in M$ do not appear in any other edge, so they are all still active when f is traversed). This yields a natural coupling with the property that

$$G_i + \mathcal{M} \subseteq G_{i+m} \quad \text{and} \quad G_{i+|\mathcal{M}|} \subseteq G_i + \mathcal{E}_{m,\mathcal{U}_i} \tag{2.5}$$ eq:cpl

for any induced matching $\mathcal{M} \subseteq \mathcal{E}_{m,\mathcal{U}_i}$.

We first prove prove (2.3), and say that f_j is *induced* if it (i) contains k distinct vertices and (ii) is vertex disjoint from $(f_h)_{1\leqslant h\leqslant m, h\neq j}$. Let $\mathcal{M}$ contain all induced edges in $\{f_1, \dots, f_m\}$. It routinely follows that

$$\mathbb{E}|\mathcal{M}| = \textstyle\sum_{j\in[m]} \mathbb{P}(f_j \text{ is good}) \geqslant m \cdot \big(1 - k(km-1)/|\mathcal{U}_i|\big) > 3m/4. \tag{2.6}$$ eq:prgood

Note that $(f_j)_{1\leqslant j\leqslant m}$ is constructed by km independent random variables, each corresponding to a (random) vertex choice from $\mathcal{U}_i$. Furthermore, changing the outcome of one vertex choice can change $|\mathcal{M}|$ by at most two. Hence a standard application of the bounded differences inequality [35] shows that

$$\mathbb{P}(|\mathcal{M}| \leqslant \lfloor m/2 \rfloor) \leqslant \mathbb{P}(|\mathcal{M}| \leqslant \mathbb{E}|\mathcal{M}| - m/4) \leqslant e^{-\lambda m} \tag{2.7}$$ eq:sandwich:bdi

for $\lambda := 1/(128k)$, say, which together with (2.5) establishes (2.3).

We now prove (2.4) by a rerandomization argument, using a permutation $\pi : \mathcal{U}_i \to \mathcal{U}_i$ that is chosen independently and uniformly at random. The key observation is that we can equivalently construct $(f_j)_{1\leqslant j\leqslant m}$ by first generating the edges $(g_j)_{1\leqslant j\leqslant m}$ with the same distribution as $(f_j)_{1\leqslant j\leqslant m}$, and then setting $f_j := \pi(g_j)$ for all $1 \leqslant j \leqslant m$, where $\pi(g) := \{\pi(w) : w \in g\}$. By (2.7) we know that, with probability at least $1 - e^{-\lambda m}$, the set $\{g_1, \dots, g_m\}$ contains at least $\lceil m/2 \rceil$ induced edges, say $\tilde{g}_1, \dots, \tilde{g}_{\lceil m/2 \rceil}$. The crux is that, even without knowing the permutation π, using knowledge of $(g_j)_{1\leqslant j\leqslant m}$ we can already conclude that all edges in

$$\mathcal{M} := \big\{\pi(\tilde{g}_1), \dots, \pi(\tilde{g}_{\lceil m/2 \rceil})\big\} \subseteq \{f_1, \dots, f_m\} = \mathcal{E}_{m,\mathcal{U}_i} \tag{2.8}$$ eq:cM:definition

are again induced. Moreover, since π is a random permutation of $\mathcal{U}_i$, the edges in $\mathcal{M}$ form a random matching with the same uniform distribution as $\mathcal{M}_{\lceil m/2 \rceil,\mathcal{U}_i}$, which together with (2.5) establishes (2.4). □

We remark that for $m = \Theta(\varepsilon|\mathcal{U}_i|)$ the above argument easily allows us to improve the number of added hyperedges from $\lceil m/2 \rceil$ to $(1 - O(\varepsilon))m$ for small ε (see (2.6)–(2.7) above).

2.2 Random evolution from initial hypergraph

lution

In view (2.3)–(2.4), the next step in the proof of Theorem 2.1 is to study the random evolution starting from a fixed initial hypergraph F. One key difference to previous related results [47, 32, 41] is that in Lemma 2.3 we only add random (matching) edges to a subset $W \subseteq V(F)$ of the vertices. Similar to Theorem 2.1, our main technical assumption is that the component distribution of F has exponentially decaying tails. The conditions in both cases of Lemma 2.3 are natural: they ensure that associated component exploration processes are subcritical or supercritical, see (2.9) and (2.10) below. Note that $S(G_i, [n]) = S(i)$ and $S(G_i, \mathcal{U}_i) = S_u(i)$, and recall that $F + \mathcal{E} = (V(F),\, E(F) \cup \mathcal{E})$.

em:evo

Lemma 2.3. *Given $k \geqslant 2$, $\beta > 1$ and $B, \xi, \gamma, \pi > 0$, there are $\alpha > 1$ and $A, \lambda, c, n_0 > 0$ such that, for all $n \geqslant n_0$, $\ell \geqslant 0$, the following holds for every n-vertex k-uniform hypergraph F with $\sum_{j\in[n]} \beta^j N_j(F) \leqslant Bn$, and every vertex-subset $W \subseteq V(F)$ of size $|W| \geqslant \xi n$, writing $S(F, W) := \sum_{w\in W} |C_w(F) \cap W|/|W|$.*

1. (Subcritical case: adding random edges) *If $k(k-1) \cdot \ell/|W| \cdot S(F, W) \leqslant 1 - \gamma$, then, with probability at least $1 - n^{-\pi}$, we have $\sum_{j\in[n]} \alpha^j N_{\geq j}(H_\ell) \leqslant An$ for $H_\ell := F + \mathcal{E}_{\ell,W}$.*
2. (Supercritical case: adding a random matching) *If $k(k-1) \cdot \ell/|W| \cdot \big(S(F, W) - 1\big) \geqslant 1 + \gamma$ and $\ell \leqslant |W|/k$, then, with probability at least $1 - e^{-\lambda n}$, we have $L_1(M_\ell) \geqslant cn$ for $M_\ell := F + \mathcal{M}_{\ell,W}$.*

The subcritical proof strategy is to consider a 'breadth first search' exploration process of $H_\ell = F + \mathcal{E}_{\ell,W}$, which iteratively finds new neighbors in W via edges of $\mathcal{E}_{\ell,W}$. Starting from an initial vertex $v_0 \in W$, this process eventually uncovers $C_{v_0}(H_\ell)$, i.e., the component of H_ℓ containing v_0. Intuitively, each vertex $v \in W$ has on average about $k|\mathcal{E}_{\ell,W}|/|W|$ adjacent edges $f_v = \{v, w_1, \dots, w_{k-1}\}$ in $\mathcal{E}_{\ell,W}$, and via each such edge f_v we find roughly $\sum_{1\leqslant h<k} |C_{w_h}(F) \cap W|$ new vertices in W. Since the vertices $w_h \in W$ are chosen uniformly at random, we expect that, on average, the exploration process finds in each step about

$$\frac{k|\mathcal{E}_{\ell,W}|}{|W|} \cdot \sum_{1\leqslant h<k} \sum_{w_h \in W} \frac{|C_{w_h}(F) \cap W|}{|W|} = \frac{k\ell}{|W|} \cdot (k-1) \cdot S(F,W) \leqslant 1 - \gamma \tag{2.9}$$

eq:evo:sub:rate

new vertices in W. By standard heuristics this suggests that the exploration process is 'subcritical', i.e., should quickly terminate (and thus should only find few vertices). Unfortunately, this reasoning is not yet enough, since the exploration process is only subcritical when restricted to vertices from $W \subseteq V(F)$. Here the saving idea is that, by the assumption that F has exponential tails, an average exploration step should only pick up additional $O(1)$ vertices outside W (which are not touched by any edges of $\mathcal{E}_{\ell,W}$, and thus are irrelevant for the termination of the exploration process). This approach eventually establishes the subcritical case by careful branching process arguments; we defer the details to Appendix C.1.

The supercritical proof strategy is similar but simpler. Again the main idea is to study the growth of a component exploration process of $M_\ell = F + \mathcal{M}_{\ell,W}$, which iteratively finds new neighbors in W via matching edges in $\mathcal{M}_{\ell,W}$. One key difference to the subcritical case is that here each vertex is contained in at most one matching edge (so we need to adjust the number of 'newly found' vertices by -1). Mimicking (2.9) we thus loosely expect that, on average, the exploration process finds in each step about

$$\frac{k|\mathcal{M}_{\ell,W}|}{|W|} \cdot \sum_{1\leqslant h<k} \sum_{w_h \in W} \frac{(|C_{w_h}(F) \cap W| - 1)}{|W|} = \frac{k\ell}{|W|} \cdot (k-1) \cdot \big(S(F,W) - 1\big) \geqslant 1 + \gamma \tag{2.10}$$

eq:evo:super:rate

new vertices in W that can still participate in matching edges. By standard heuristics this suggests that the exploration process is 'supercritical', i.e., should find a giant component. This approach eventually establishes the subcritical case by fairly routine random walk arguments; we defer the details to Appendix C.2.

2.3 Proof of rough control result

:proof

Finally, Theorem 2.1 follows easily from Lemma 2.2 and 2.3; we include the details for completeness.

Proof of Theorem 2.1. We start with inequality (2.1). Applying the subcritical case of Lemma 2.3 with $F = G_i$, $W = \mathcal{U}_i$, $S(F,W) = S_u(i)$, $\ell = 2m$, $\gamma = 1/2$ and π set to $\pi+1$, noting $k(k-1)\cdot 2m/|\mathcal{U}_i| \cdot S_u(i) \leqslant 1/2$ and $|\mathcal{U}_i| \geqslant 8k^2 m > \xi n$ it follows that there are constants $\alpha > 1$ and $A, n_1 > 0$ such that

$$\Pi := \mathbb{P}\big(\textstyle\sum_{j\in[n]} \alpha^j N_{\geqslant j}(G_i + \mathcal{E}_{2m,\mathcal{U}_i}) \geqslant An \mid G_i\big) \leqslant n^{-(\pi+1)}$$

for $n \geqslant n_1$. Applying (2.3) of Lemma 2.2 with m set to $2m$, noting $2m \leqslant U(i)/(4k^2)$ it follows that there is a constant $\lambda > 0$ such that monotonicity of $N_{\geqslant j}(\cdot)$ yields

$$\mathbb{P}\big(\textstyle\sum_{j\in[n]} \alpha^j N_{\geqslant j}(G_{i+m}) \geqslant An \mid G_i\big) \leqslant \Pi + e^{-\lambda m} \leqslant n^{-(\pi+1)} + e^{-\lambda\xi n},$$

which establishes inequality (2.1) for $n \geqslant n_0(\pi, \lambda, \xi, n_1)$ large enough.

For inequality (2.2) we proceed similarly. Noting that $k(k-1)\cdot\lceil m/2\rceil/|\mathcal{U}_i| \cdot (S_u(i)-1) \geqslant 2$, here we shall apply the supercritical case of Lemma 2.3 with $F = G_i$, $W = \mathcal{U}_i$, $S(F,W) = S_u(i)$, $\ell = \lceil m/2\rceil$ and $\gamma = 1$. Together with (2.4) of Lemma 2.2, it follows that there are constants $c, \lambda, n_1 > 0$ such that

$$\mathbb{P}\big(L_1(G_{i+m}) \leqslant cn \mid G_i\big) \leqslant \mathbb{P}\big(L_1(G_i + \mathcal{M}_{\lceil m/2\rceil,\mathcal{U}_i}) \leqslant cn \mid G_i\big) + e^{-\lambda m} \leqslant e^{-\lambda n} + e^{-\lambda\xi n}$$

for $n \geqslant n_1$. This establishes inequality (2.2) with $\sigma := \lambda \min\{1, \xi\}/2$ for $n \geqslant n_0(\lambda, \xi, n_1)$. □

3 Phase transition: emergence of giant component

:proof

In this core section we prove the phase transition result Theorem 1.6, and in view of Remark 1.8 we shall again consider the multigraph variant $(G_i)_{i\geqslant 0} = (G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$ of the k-uniform $\mathbf{d}_n$-process. As indicated in Section 1.3, our approach requires numerous auxiliary random variables. Let $X_{a,b}(i)$ denote the number of vertices $v \in [n]$ of G_i with degree a and degree restriction b, i.e., with $\deg_{G_i}(v) = a$ and $d_v^{(n)} = b$. The assumptions of Theorem 1.6 allow us to restrict our attention to degree pairs $(a, b) \in \mathcal{V}$, where

$$\mathcal{V} := \big\{(x,y) \in \mathbb{N}^2 \,:\, 0 \leqslant x \leqslant y \leqslant \Delta\big\} \qquad \text{and} \qquad \mathcal{V}^* := \big\{(x,y) \in \mathcal{V} \,:\, x < y\big\}. \tag{3.1}$$

def:cVcVS

Writing $(C_j)_{j\in\mathfrak{C}_i}$ for the components of G_i, the key susceptibility related random variables are

$$Z_{a,b,c,d}(i) := \sum_{j\in\mathfrak{C}_i} |C_{j,a,b}||C_{j,c,d}| \qquad \text{for all } (a,b),(c,d) \in \mathcal{V}, \tag{3.2}$$

eq:Z:def

where the (degree based) component subsets $C_{j,a,b} \subseteq C_j$ are given by

$$C_{j,a,b} := \big\{v \in C_j \,:\, \deg_{C_j}(v) = a \text{ and } d_v^{(n)} = b\big\}. \tag{3.3}$$

eq:Cjab:def

Since $|C_j| = \sum_{(a,b)\in\mathcal{V}} |C_{j,a,b}|$, the variables $Z_{a,b,c,d}(i)$ refine the susceptibility (1.1) in view of

$$S(i) = \sum_{w\in[n]} \frac{|C_w(G_i)|}{n} = \sum_{j\in\mathfrak{C}_i} \frac{|C_j|^2}{n} = \sum_{(a,b),(c,d)\in\mathcal{V}} \frac{Z_{a,b,c,d}(i)}{n}. \tag{3.4}$$

def:SiZ

Since $|C_j \cap \mathcal{U}_i| = \sum_{(a,b)\in\mathcal{V}^*} |C_{j,a,b}|$, they similarly refine the active susceptibility (1.4) in view of

$$S_u(i) = \sum_{w\in\mathcal{U}_i} \frac{|C_w(G_i) \cap \mathcal{U}_i|}{|\mathcal{U}_i|} = \sum_{j\in\mathfrak{C}_i} \frac{|C_j \cap \mathcal{U}_i|^2}{|\mathcal{U}_i|} = \sum_{(a,b),(c,d)\in\mathcal{V}^*} \frac{Z_{a,b,c,d}(i)}{U(i)}, \tag{3.5}$$

def:SuiZ

where $U(i) := |\mathcal{U}_i|$ denotes the number of active vertices $v \in [n]$ in G_i, i.e., with $(\deg_{G_i}(v), d^{(n)}) \in \mathcal{V}^*$. As we shall see in Section 3.3.2, the key conceptual point is that the extra auxiliary variables $Z_{a,b,c,d}(i)$ allow us to obtain a 'closed' system of random variables, i.e., where we can estimate the expected one-step changes of all variables using only variables in the system (which would fail otherwise, as indicated in Section 1.3.1).

The following technical result is at the heart of our inductive proof approach: it relates the evolution of the degree restricted process with suitable differential equations. In particular, the approximations (3.7)–(3.9) show that typically $U(tn) \approx u(t)n$, $X_{a,b}(tn) \approx x_{a,b}(t)n$ and $Z_{a,b,c,d}(tn) \approx z_{a,b,c,d}(t)n$ for $0 \leqslant t \leqslant t_h$, where the deterministic functions $u(t)$, $x_{a,b}(t)$ and $z_{a,b,c,d}(t)$ are the solutions to a system of $O(\Delta^4)$ differential equations. By Remark 3.3 these approximations also ensure $S(tn) \approx s(t)$ and $S_u(tn) \approx s_u(t)$, where the functions $s(t)$ and $s_u(t)$ are defined in terms of $u(t)$ and $z_{a,b,c,d}(t)$. The technical 'exponential tail' bound (3.10) is optimized for inductive applications of the rough control result Theorem 2.1, and by Remark 3.2 it also ensures $L_1(tn) = O(\log n)$ for $0 \leqslant t \leqslant t_h$. The definition (3.6) of the time-sequence $(t_h)_{h\geqslant 0}$ differs slightly from the heuristic form (1.10): instead of $u(t_h)$ we here use $T - t_h$ in the numerator, which makes it easier to see that $t_h < T$, among other advantages. In particular, in Section 3.1 we shall define $t_c := \lim_{h\to\infty} t_h$, and the definition (3.6) then makes it easy to see that $\lim_{h\to\infty} s_u(t_h)$ blows up when $t_c < T$.

hm:dgl

Theorem 3.1 (Main technical result)**.** *Suppose that $k \geqslant 2$, $\Delta \geqslant 1$, $\mathbf{r} = (r_0, \dots, r_\Delta) \in [0,1]^{\Delta+1}$ and $\mathbf{d}_n = \big(d_1^{(n)}, \dots, d_n^{(n)}\big) \in \{0, \dots, \Delta\}^n$ satisfy the assumptions of Theorem 1.6. Set $T := \sum_{j\in[\Delta]} jr_j/k$. Then there is a sequence $\big(t_h, \beta_h, B_h, n_{0,h}, \Psi_h(n), \xi_h(n)\big)_{h\geqslant 0}$ of constants $t_h, \beta_h, B_h, n_{0,h} \geqslant 0$ with $t_0 = 0$ and $\beta_h > 1$ and non-negative functions $\Psi_h(n), \xi_h(n)$ such that the following holds for each fixed $h \geqslant 0$:*

(P1) On $[0, t_h]$ the system of differential equations (3.13)–(3.17) has unique solutions $u(t)$, $x_{a,b}(t)$ and $z_{a,b,c,d}(t)$ for all $(a,b),(c,d) \in \mathcal{V}$. These solutions satisfy $u(t) \geqslant k(T - t_h)/(4\Delta)$ and $z_{a,b,c,d}(t) \leqslant 2B_h/(\beta_h - 1)$, and they also have bounded first derivatives.

(P2) If $h \geqslant 1$, then we have $t_{h-1} < t_h < T$, $\beta_h \leqslant \beta_{h-1}$, $B_h \geqslant B_{h-1}$ and

$$t_h - t_{h-1} = \frac{T - t_{h-1}}{32\Delta k \cdot \max\{s_u(t_{h-1}), 1\}}, \tag{3.6}$$ ind:thd

where the expression $s_u(t_{h-1})$ is well-defined by (P1) and (3.11) below.

(P3) We have $\Psi_h(n) = o(n^{-99})$ and $\xi_h(n) = o(1)$, with $\xi_h(n) \leqslant k(T - t_h)/(8\Delta)$ for $n \geqslant n_{0,h}$.

(P4) With probability at least $1 - \Psi_h(n)$, for $n \geqslant n_{0,h}$ and all $0 \leqslant i \leqslant t_h n$ we have

$$\frac{U(i)}{n} = u(i/n) \pm \xi_h(n), \tag{3.7}$$ ind:U

$$\frac{X_{a,b}(i)}{n} = x_{a,b}(i/n) \pm \xi_h(n) \qquad \text{for all } (a,b) \in \mathcal{V}, \tag{3.8}$$ ind:X

$$\frac{Z_{a,b,c,d}(i)}{n} = z_{a,b,c,d}(i/n) \pm \xi_h(n) \qquad \text{for all } (a,b),(c,d) \in \mathcal{V}, \tag{3.9}$$ ind:Z

$$\sum_{j \in [n]} \beta_h^j N_{\geqslant j}(G_i) \leqslant B_h n, \tag{3.10}$$ ind:tail

where $G_i = G_{n,i}^{k,\mathbf{d}_n}$ is the multigraph variant of the k-uniform $\mathbf{d}_n$-process.

em:dgl **Remark 3.2.** *Setting $b_h := \log \beta_h > 0$ and $C_h := (1+B_h)/b_h$, note that inequality (3.10) implies $N_{\geqslant j}(G_i) \leqslant B_h e^{-b_h j} n$, which in particular gives $L_1(G_i) \leqslant C_h \log n$ for $n \geqslant e$, say.*

rem:s **Remark 3.3.** *In view of the identities (3.4)–(3.5), using (P1) we define*

$$s(t) := \sum_{(a,b),(c,d) \in \mathcal{V}} z_{a,b,c,d}(t) \qquad \text{and} \qquad s_u(t) := \sum_{(a,b),(c,d) \in \mathcal{V}^*} z_{a,b,c,d}(t)/u(t). \tag{3.11}$$ def:stsut

Combining the inequalities from (P1) and (P3) with (3.4)–(3.5), there is $D_h = D_h(k, \Delta, T, t_h, \beta_h, B_h) > 0$ such that, for $n \geqslant n_{0,h}$ and all $0 \leqslant i \leqslant t_h n$, the approximations (3.7) and (3.9) imply

$$S(i) = s(i/n) \pm \Delta^4 \xi_h(n) \qquad \text{and} \qquad S_u(i) = s_u(i/n) \pm D_h \xi_h(n). \tag{3.12}$$ ind:SSu

As we shall see, the slightly roundabout statement of Theorem 3.1 has the advantage of only requiring very little analytical knowledge about the functions $u(t)$, $x_{a,b}(t)$ and $z_{a,b,c,d}(t)$. In fact, in the deferred proof of Theorem 3.1 the differential equation method based approximations (3.7)–(3.9) will work hand in hand with the rough control based exponential tail bound (3.10), i.e., they inductively enable each other.

The remainder of this section is organized as follows. In Section 3.1 we state the relevant system of differential equations, define the critical point t_c, and show that $s_u(t)$ blows at t_c. Using these properties, in Section 3.2 we then prove Theorem 1.6, by combining our main technical result Theorem 3.1 with the rough control result Theorem 2.1. Finally, Sections 3.3–3.4 are devoted to the deferred proof of Theorem 3.1.

3.1 Differential equations and definition of t_c

ec:ODE

As suggested by Theorem 3.1, for given $k \geqslant 2$, $\Delta \geqslant 1$ and $\mathbf{r} = (r_0, \dots, r_\Delta) \in [0,1]^{\Delta+1}$ we are interested in the solution to the following system[6] of differential equations (heuristically derived in Section 3.3):

$$u'(t) = -\frac{k \sum_{b \in [\Delta]} x_{b-1,b}(t)}{u(t)}, \tag{3.13}$$ eq:U:diff

$$x'_{a,b}(t) = \frac{k\big(\mathbb{1}_{\{a>0\}} x_{a-1,b}(t) - \mathbb{1}_{\{a<b\}} x_{a,b}(t)\big)}{u(t)}, \tag{3.14}$$ eq:X:diff

$$\begin{aligned} z'_{a,b,c,d}(t) = {} & \frac{k(k-1) \sum_{(e,f) \in \mathcal{V}^*} \Big(z_{a,b,e,f}(t) + \delta_{a,b}^{e,f} x_{e,f}(t) \Big) \sum_{(e,f) \in \mathcal{V}^*} \Big(z_{e,f,c,d}(t) + \delta_{c,d}^{e,f} x_{e,f}(t) \Big)}{u(t)^2} \\ & + \frac{k \sum_{(e,f) \in \mathcal{V}^*} \Big(z_{a,b,e,f}(t) \delta_{c,d}^{e,f} + \delta_{a,b}^{e,f} z_{e,f,c,d}(t) + \delta_{a,b}^{e,f} \delta_{c,d}^{e,f} x_{e,f}(t) \Big)}{u(t)}, \end{aligned} \tag{3.15}$$ eq:Z:diff

[6] Introducing the technically redundant function $u(t) = \sum_{b \in [\Delta]} \big(x_{0,b}(0) - x_{b,b}(t) \big)$ allows for cleaner equations.

for all $(a,b),(c,d) \in \mathcal{V}$, with initial conditions

$$u(0) = \sum_{b\in[\Delta]} r_b, \qquad x_{a,b}(0) = \mathbb{1}_{\{a=0\}} r_b \qquad \text{and} \qquad z_{a,b,c,d}(0) = \mathbb{1}_{\{a=c=0,\, b=d\}} r_b, \tag{3.16}$$

eq:UXZ:init

where in (3.15) we tacitly used the shorthand

$$\delta^{e,f}_{x,y} := \begin{cases} 1, & \text{if } y = f \text{ and } x = e+1, \\ -1, & \text{if } y = f \text{ and } x = e, \\ 0, & \text{otherwise.} \end{cases} \tag{3.17}$$

def:delta

Our proof framework avoids the usual technical analysis of this large system of differential equations, by effectively transferring properties of the degree restricted process to the solutions of (3.13)–(3.17).

or:fkt **Corollary 3.4.** *For each $h \geqslant 1$, the following holds on $[0,t_h]$: the function $s(t)$ is monotone increasing, and we have $z_{a,b,c,d}(t) \geqslant 0$, $u(t) \leqslant 1$ and $\min\{s(t), s_u(t)\} \geqslant 1$.*

Proof. By definition $Z_{a,b,c,d}(i) \geqslant 0$, so the approximation (3.9) implies that whp $0 \leqslant Z_{a,b,c,d}(\lfloor tn \rfloor)/n \leqslant z_{a,b,c,d}(t) + o(1)$ for all $t \in [0,t_h]$. This implies $z_{a,b,c,d}(t) \geqslant 0$ on $[0,t_h]$, since the function $z_{a,b,c,d}(t)$ is defined without reference to n. Noting $U(i) = |\mathcal{U}_i| \leqslant n$ as well as $S(i) \geqslant \sum_{w\in[n]} 1/n = 1$ and $S_u(i) \geqslant \sum_{w\in[n]} \mathbb{1}_{\{w\in\mathcal{U}_i\}}/|\mathcal{U}_i| = 1$, using (3.7) and (3.12) we similarly obtain $u(t) \leqslant 1$ and $\min\{s(t), s_u(t)\} \geqslant 1$. Since $S(i) = \sum_{w\in[n]} |C_w(G_i)|/n$ is monotone increasing in each step, using the inequality $S(i+j) \geqslant S(i)$ together with (3.12) we here obtain $s(\tau_2) \geqslant s(\tau_1)$ for all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_h$, i.e., that $s(t)$ is monotone increasing. □

In particular, while previous related work [4, 14, 47, 32, 11] needed technical analysis to establish existence of a blow-up point t_c of the relevant differential equations (which for (3.13)–(3.17) would require some care due to the unusually large number of $O(\Delta^4)$ variables), we can simply define the critical point as

$$t_c := \lim_{h\to\infty} t_h, \tag{3.18}$$

def:tc

where the explicit construction (3.6) of the increasing times $(t_h)_{h\geqslant 0}$ then nearly automatically guarantees that the idealized susceptibility $s(t)$ and active susceptibility $s_u(t)$ both blow up at t_c, provided that $t_c < T$ (the fact that this works can ultimately be traced back to the rough control result Theorem 2.1, i.e., to combinatorial and probabilistic properties of the degree restricted process; cf. Section 3.4). In fact, for our purposes it suffices to show that $s_u(t_h)$ blows up as $h \to \infty$, which is easier to establish; cf. Lemma 5.1

cor:tc **Corollary 3.5.** *We have $t_c = \sup_{h\geqslant 1} t_h \in (0,T]$. Furthermore, $t_c < T$ implies $\lim_{t\nearrow t_c} s(t) \to \infty$ and $\lim_{h\to\infty} s_u(t_h) = \infty$.*

Proof. By monotone convergence, $t_c = \sup_{h\geqslant 1} t_h \in (0,T]$ is immediate (since $0 = t_0 < t_h < T$ for $h \geqslant 1$). Using $s_u(t_{h-1}) \geqslant 1$ and $t_h \leqslant t_c$, by rearranging (3.6) we infer $s_u(t_{h-1}) \geqslant (32k\Delta)^{-1}(T - t_c)/(t_c - t_{h-1})$ for all $h \geqslant 1$. Note that $z_{a,b,c,d}(t) \geqslant 0$ and (P1) similarly imply $s(t_h) \geqslant u(t_h)s_u(t_h) \geqslant k(T-t_c)/(4\Delta)\cdot s_u(t_h)$ for all $h \geqslant 0$. If $t_c < T$, then $\lim_{h\to\infty} s_u(t_h) = \infty$, and monotonicity of $s(t)$ implies $\lim_{t\nearrow t_c} s_u(t) = \infty$. □

3.2 Proof of main phase transition result

c:DGL2

In this subsection we prove Theorem 1.6 following the outline from Section 1.3.2, by combining our main technical result Theorem 3.1 with the properties of Section 3.1 and the rough control result Theorem 2.1. As mentioned in Remark 1.8, to this end it suffices to consider the multigraph variant $(G_i)_{i\geqslant 0} = (G^{k,\mathbf{d}_n}_{n,i})_{i\geqslant 0}$.

3.2.1 Theorem 1.6: subcritical phase and susceptibility

ec:sub

Proof of the 'subcritical phase' and 'susceptibility' part of Theorem 1.6. Given $0 \leqslant t < t_c$, we use the conclusions of Theorem 3.1 for the smallest $h = h(t) \geqslant 1$ satisfying $t_h \geqslant t$. Recalling Remark 1.8, this readily completes the proof by combining the tail bounds of Remark 3.2 and the susceptibility approximation (3.12) with $\Psi_h(n) = o(n^{-99})$, $s(t) \geqslant 1$ and Corollary 3.5 (setting $C := C_h$, $\xi_n := \Delta^4 \xi_h(n)$, $a := b_h$ and $A := B_h$). □

3.2.2 Theorem 1.6: supercritical phase and giant component

:super

Proof of the 'supercritical phase' part of Theorem 1.6. Since $t_c < t < T$, we may henceforth assume $t_c < T$. Using Corollary 3.5, we therefore can pick $h \geqslant 1$ large enough (and thus $s_u(t_h)$ large enough) such that

$$\tau_h := \min\left\{\frac{4}{k(k-1)\cdot\max\{s_u(t_h)-2,1\}},\ \frac{1}{64k^2},\ \frac{t-t_c}{4}\right\} = \frac{4}{k(k-1)\cdot(s_u(t_h)-2)}. \tag{3.19}$$

eq:tauh:super

We now define $\mathcal{E}_h$ as the event that G_{i_h} satisfies (3.7)–(3.10) for $i = i_h$, where

$$i_h := \lfloor t_h n\rfloor \quad \text{and} \quad m_h := \lfloor 4\tau_h u(t_h) n\rfloor. \tag{3.20}$$

def:ih:mh

Noting that Theorem 3.1 implies $\mathbb{P}(\neg\mathcal{E}_h) \leqslant \Psi_h(n) = o(n^{-99})$, we henceforth condition on G_{i_h}, and assume that $\mathcal{E}_h$ holds. Using $t_h \leqslant t_c$ and $u(t_h) \leqslant 1$, by definition (3.19) of τ_h it follows that

$$i_h + m_h \leqslant \big(t_h + 4\tau_h u(t_h)\big)n \leqslant (t_c + 4\tau_h)n \leqslant tn. \tag{3.21}$$

eq:m:sup

The approximations (3.7) and (3.11) imply $S_u(i_h) \geqslant s_u(t_h) - 1$ and $u(t_h)n/2 \leqslant |\mathcal{U}_{i_h}| \leqslant 2u(t_h)n$ for large n. By definition (3.19) of τ_h, it follows that $m_h \leqslant 8\tau_h|\mathcal{U}_{i_h}| \leqslant |\mathcal{U}_{i_h}|/(8k^2)$ and

$$k(k-1)\cdot m_h/|\mathcal{U}_{i_h}|\cdot\big(S_u(i_h)-1\big) \geqslant k(k-1)\cdot\tau_h\cdot\big(s_u(t_h)-2\big) = 4. \tag{3.22}$$

eq:tail:sup

Since G_{i_h} satisfies (3.10) for $i = i_h$, by invoking inequality (2.2) from Theorem 2.1 with $i = i_h$, $m = m_h$, $\beta = \beta_h$, $B = B_h$, and $\xi = \tau_h u(t_h)$, there are constants $c, \sigma > 0$ such that

$$\mathbb{P}\big(L_1(G_{\lfloor tn\rfloor}) \leqslant cn \mid G_{i_h}\big) \leqslant \mathbb{P}\big(L_1(G_{i_h+m_h}) \leqslant cn \mid G_{i_h}\big) \leqslant O(e^{-\sigma n}) = o(n^{-99}), \tag{3.23}$$

eq:mainhg:L1

which by recalling $\mathbb{P}(\neg\mathcal{E}_h) \leqslant \Psi_h(n) = o(n^{-99})$ and Remark 1.8 then completes the proof. □

3.2.3 Theorem 1.6: upper bound on t_c

gcsuff

To complete the proof of Theorem 1.6, it remains to show that the critical point satisfies $t_c \leqslant 1/(k-1)$. Our main tool is Lemma 3.6, which is based on the following basic heuristic: as long as all components have size $o(n)$, each step should join k distinct components with probability close to one, which suggests that a giant component should appear after roughly $n/(k-1)$ steps (see [44, 39, 40] for related arguments).

gcsuff

Lemma 3.6. *Suppose that the assumptions of Theorem 1.6 hold. For every $t \in \big(1/(k-1), T\big)$ there are $\gamma, \lambda > 0$ depending only on t, T, k, Δ such that $\mathbb{P}(L_1(G_{\lfloor tn\rfloor}) \geqslant \gamma n) = 1 - O(e^{-\lambda n})$.*

Proof of the $t_c \leqslant 1/(k-1)$ bound of Theorem 1.6. Aiming at a contradiction, suppose that $t_c > 1/(k-1)$. Pick $1/(k-1) < t < t_c$, and recall that $t_c \leqslant T$. Applying Lemma 3.6 and the subcritical part of the multigraph variant of Theorem 1.6 (as proved in Section 3.2.1), it follows that whp $\gamma n \leqslant L_1(G_{\lfloor tn\rfloor}) \leqslant C\log n$, which for large n yields the desired contradiction. □

Proof of Lemma 3.6. Set $m := \lfloor tn\rfloor$. Pick $\varepsilon > 0$ small enough such that $t(k-1)\cdot(1-\varepsilon)^{k-1} \geqslant 1 + 2\varepsilon$. Let $\mathcal{B}_j$ denote the event that, for all $0 \leqslant i \leqslant j$, any component of G_i contains at most $\varepsilon|\mathcal{U}_i|/(k-1)$ active vertices. We call a step successful if k distinct components are joined up, and define $\mathcal{S}_m$ as the event that at least $n/(k-1)$ of the first i steps are successful. The point is that if $\mathcal{B}_i$ holds, then the next step $i+1$ is successful with probability at least $(1-\varepsilon)^{k-1}$, say. Since $\lfloor tn\rfloor\cdot(1-\varepsilon)^{k-1} \geqslant (1+\varepsilon)\cdot n/(k-1)$ for large n, using standard Chernoff bounds (and stochastic domination) it routinely follows that $\mathbb{P}(\neg\mathcal{S}_m \cap \mathcal{B}_m) = O(e^{-\lambda n})$ for suitable $\lambda = \lambda(\varepsilon, t, k) > 0$. It remains to show that $\mathcal{S}_m \cup \neg\mathcal{B}_m$ implies $L_1(G_m) \geqslant \gamma n$ for some $\gamma = \gamma(\varepsilon, t, T, \Delta) > 0$. If $\mathcal{B}_m$ fails, then $L_1(G_i) \geqslant \varepsilon|\mathcal{U}_i|/k$ for some step $i \leqslant m$, so by combining $|\mathcal{U}_i|\Delta \geqslant \sum_{v\in[n]}(d_v^{(n)} - \deg_{G_i}(v))$ and $\sum_{v\in[n]}\deg_{G_i}(v) \leqslant ki \leqslant ktn$ with $\sum_{v\in[n]} d_v^{(n)}/(kn) \to T$ (see above Theorem 1.6) it follows that

$$\frac{L_1(G_m)}{n} \geqslant \frac{\varepsilon|\mathcal{U}_i|}{kn} \geqslant \frac{\varepsilon\big(\sum_{v\in[n]} d_v^{(n)}/(kn) - t\big)}{\Delta} \geqslant \frac{\varepsilon(T-t)}{2\Delta} =: \gamma \tag{3.24}$$

eq:gcsuff

for large n. Furthermore, since each successful step reduces the number of components by $k-1$, it also follows that $\mathcal{S}_m$ implies $L_1(G_m) = n \geqslant \gamma n$, completing the proof. □

3.3 Analysis of variables: initial values and one-step changes

ec:DGL

In this preparatory subsection we lay the groundwork for the upcoming proof of Theorem 3.1, which hinges on an application of the differential equation method [54, 53, 51] to the $O(\Delta^4)$ random variables $U(i)$, $X_{a,b}(i)$ and $Z_{a,b,c,d}(i)$. In particular, tacitly assuming that the assumptions of Theorem 1.6 hold, we investigate the three main conditions of the differential equation method (trend hypothesis, boundedness hypothesis, and initial condition), and heuristically motivate the relevant system of differential equations (3.13)–(3.17).

3.3.1 Degree related variables

:basic

We start by analyzing the degree variables $X_{a,b}(i)$, tacitly assuming $(a,b) \in \mathcal{V}$. Using assumption (1.15), it follows that the initial values of $X_{a,b}(i)$ satisfy the *initial condition*

$$\frac{X_{a,b}(0)}{n} = \frac{\mathbb{1}_{\{a=0\}}\big|\big\{v \in [n] : d_v^{(n)} = b\big\}\big|}{n} = \mathbb{1}_{\{a=0\}} r_b \pm \lambda_0(n) \qquad \text{with } \lambda_0(n) = o(1), \tag{3.25}$$

eq:X:start

where $\lambda_b(n) \geqslant 0$ depends on $\mathbf{d}_n$ and $\mathbf{r}$. Since in each step the degrees of at most k vertices are altered, it follows that the maximum one-step changes of $X_{a,b}(i)$ satisfy the *boundedness hypothesis*

$$\big|X_{a,b}(i+1) - X_{a,b}(i)\big| \leqslant k = O(1). \tag{3.26}$$

eq:X:bound

In each step, with probability at least $1 - k^2/|\mathcal{U}_i|$ all k randomly chosen active vertices $v_{i+1,1}, \dots, v_{i+1,k} \in \mathcal{U}_i$ are distinct. With the worst-case changes from (3.26) and $|\mathcal{U}_i| = U(i)$ in mind, similarly to [54, 53, 45] it now routinely follows that the expected one-step changes of $X_{a,b}(i)$ satisfy the *trend hypothesis*

$$\begin{aligned} \mathbb{E}(X_{a,b}(i+1) - X_{a,b}(i) \mid G_i) &= \sum_{1 \leqslant j \leqslant k} \frac{\mathbb{1}_{\{a>0\}} X_{a-1,b}(i) - \mathbb{1}_{\{a<b\}} X_{a,b}(i)}{|\mathcal{U}_i|} + O\Big(\frac{1}{|\mathcal{U}_i|}\Big) \\ &= \frac{k\big(\mathbb{1}_{\{a>0\}} X_{a-1,b}(i) - \mathbb{1}_{\{a<b\}} X_{a,b}(i)\big)}{U(i)} + O\Big(\frac{1}{U(i)}\Big). \end{aligned} \tag{3.27}$$

eq:X:change

We next analyze the number of active vertices $U(i) = |\mathcal{U}_i| = \sum_{b \in [\Delta]} \big(X_{0,b}(0) - X_{b,b}(i)\big)$. Using estimate (3.25), it follows that the initial value of $U(i)$ satisfies the *initial condition*

$$\frac{U(0)}{n} = \frac{\sum_{b \in [\Delta]} X_{0,b}(0)}{n} = \sum_{b \in [\Delta]} r_b \pm \Delta\lambda_0(n). \tag{3.28}$$

eq:U:start

Analogously to (3.26)–(3.27), the one-step changes of $U(i)$ satisfy the *boundedness and trend hypothesis*

$$\big|U(i+1) - U(i)\big| \;\leqslant\; k = O(1), \tag{3.29}$$

eq:U:bound

$$\mathbb{E}(U(i+1) - U(i) \mid G_i) \;=\; -\frac{k \sum_{b \in [\Delta]} X_{b-1,b}(i)}{U(i)} + O\Big(\frac{1}{U(i)}\Big). \tag{3.30}$$

eq:U:change

To motivate the differential equations (3.13)–(3.16) for $u(t)$ and $x_{a,b}(t)$, we assume the deterministic approximations $U(i) \approx u(t)n$ and $X_{a,b}(i) \approx x_{a,b}(t)n$ with $t = i/n$. By inserting these into both sides of the expected one-step changes (3.30), noting $U(i+1) - U(i) \approx [u(t+1/n) - u(t)]n \approx u'(t)$ we anticipate $u'(t) = -k \sum_{b \in [\Delta]} x_{b-1,b}(t)/u(t)$ as in (3.13). The initial value (3.28) also suggests $u(0) = \sum_{b \in [\Delta]} r_b$ as in (3.16). Using (3.27) and (3.25) we similarly anticipate the derivative (3.14) and initial value (3.16) of $x_{a,b}(t)$.

3.3.2 Susceptibility variables

c:DE:s

We now analyze the susceptibility related key variables $Z_{a,b,c,d}(i)$, tacitly assuming $(a,b),(c,d) \in \mathcal{V}$. In G_0 all component have size one, so that $|C_{j,a,b}||C_{j,c,d}| \in \{0,1\}$, with $|C_{j,a,b}||C_{j,c,d}| = 0$ unless $a = c$ and $b = d$. Using estimate (3.25), it follows that the initial values of $Z_{a,b,c,d}(i)$ satisfy the *initial condition*

$$\frac{Z_{a,b,c,d}(0)}{n} = \frac{\mathbb{1}_{\{a=c,\, b=d\}} X_{a,b}(0)}{n} = \mathbb{1}_{\{a=c=0,\, b=d\}} r_b \pm \lambda_0(n). \tag{3.31}$$

eq:Z:start

Since in each step at most k components are joined, it follows that the maximum one-step changes of $Z_{a,b,c,d}(i)$ satisfy the *boundedness hypothesis*

$$\big|Z_{a,b,c,d}(i+1) - Z_{a,b,c,d}(i)\big| \leqslant kL_1(i)^2 + \big(kL_1(i)\big)^2 = O\big(L_1(i)^2\big). \tag{3.32}$$ eq:Z:bound

For the expected one-step changes of $Z_{a,b,c,d}(i)$, we first suppose that k distinct components $C_{j_1}, \dots, C_{j_k}$ of G_i are joined via the edge $e_{i+1} = \{v_{i+1,1}, \dots, v_{i+1,k}\}$, where $v_{i+1,\ell} \in C_{j_\ell, e_\ell, f_\ell}$ satisfies $(e_\ell, f_\ell) \in \mathcal{V}^*$ and thus $v_{i+1,\ell} \in \mathcal{U}_i$. Denoting the resulting component by C_π^+, for all degree pairs $(x,y) \in \mathcal{V}$ the number of vertices $v \in C_\pi^+$ with $\deg_{C_\pi^+}(v) = x$ and $d_v^{(n)} = y$ are then given by

$$|C_{\pi,x,y}^+| = \sum_{\ell\in[k]} \Big(|C_{j_\ell,x,y}| + \delta_{x,y}^{e_\ell,f_\ell} \Big), \tag{3.33}$$ eq:def:Cpi:xy

where the shorthand $\delta_{x,y}^{e,f} \in \{-1,0,1\}$ is defined as in (3.17), see Section 3.1. In each step, with probability at least $1 - k^2 L_1(i)/|\mathcal{U}_i|$ all k randomly chosen active vertices $v_{i+1,1}, \dots, v_{i+1,k} \in \mathcal{U}_i$ are in distinct components. With the worst case changes (3.32) in mind, by our above discussion it now follows that

$$\begin{aligned}
&\mathbb{E}(Z_{a,b,c,d}(i+1) - Z_{a,b,c,d}(i) \mid G_i) \\
&\quad = \sum_{\substack{j_1\in\mathfrak{C}_i \\ (e_1,f_1)\in\mathcal{V}^*}} \cdots \sum_{\substack{j_k\in\mathfrak{C}_i \\ (e_k,f_k)\in\mathcal{V}^*}} \Big(|C_{\pi,a,b}^+||C_{\pi,c,d}^+| - \sum_{1\leq\ell\leq k} |C_{j_\ell,a,b}||C_{j_\ell,c,d}| \Big) \prod_{1\leqslant\ell\leqslant k} \frac{|C_{j_\ell,e_\ell,f_\ell}|}{|\mathcal{U}_i|} + O\Big(\frac{L_1(i)^3}{|\mathcal{U}_i|}\Big),
\end{aligned}$$

where the $|C_{\pi,x,y}^+|$ are here *defined* by (3.33) above (which together with the additive error term formally accounts for the degenerate cases where some components C_{j_h} coincide). Recalling (3.33), we have

$$\begin{aligned}
&|C_{\pi,a,b}^+||C_{\pi,c,d}^+| - \sum_{1\leq\ell\leq k} |C_{j_\ell,a,b}||C_{j_\ell,c,d}| \\
&\quad = \sum_{\substack{h,\ell\in[k] \\ h\neq\ell}} \Big(|C_{j_h,a,b}| + \delta_{a,b}^{e_h,f_h} \Big) \Big(|C_{j_\ell,c,d}| + \delta_{c,d}^{e_\ell,f_\ell} \Big) + \sum_{1\leq\ell\leq k} \Big(|C_{j_\ell,a,b}| \delta_{c,d}^{e_\ell,f_\ell} + \delta_{a,b}^{e_\ell,f_\ell} |C_{j_\ell,c,d}| + \delta_{a,b}^{e_\ell,f_\ell} \delta_{c,d}^{e_\ell,f_\ell} \Big).
\end{aligned}$$

Since $\sum_{j_h\in\mathfrak{C}_i,(e_h,f_h)\in\mathcal{V}^*} |C_{j_h,e_h,f_h}| = \sum_{j_h\in\mathfrak{C}_i} |C_{j_h} \cap \mathcal{U}_i| = |\mathcal{U}_i|$ by (3.3), it follows that

$$\begin{aligned}
&\mathbb{E}(Z_{a,b,c,d}(i+1) - Z_{a,b,c,d}(i) \mid G_i) \\
&\quad = \sum_{\substack{h,\ell\in[k] \\ h\neq\ell}} \sum_{\substack{j_h\in\mathfrak{C}_i \\ (e_h,f_h)\in\mathcal{V}^*}} \frac{|C_{j_h,e_h,f_h}| \Big(|C_{j_h,a,b}| + \delta_{a,b}^{e_h,f_h} \Big)}{|\mathcal{U}_i|} \sum_{\substack{j_\ell\in\mathfrak{C}_i \\ (e_\ell,f_\ell)\in\mathcal{V}^*}} \frac{|C_{j_\ell,e_\ell,f_\ell}| \Big(|C_{j_\ell,c,d}| + \delta_{c,d}^{e_\ell,f_\ell} \Big)}{|\mathcal{U}_i|} \\
&\qquad + \sum_{1\leq\ell\leq k} \sum_{\substack{j_\ell\in\mathfrak{C}_i \\ (e_\ell,f_\ell)\in\mathcal{V}^*}} \frac{|C_{j_\ell,e_\ell,f_\ell}| \Big(|C_{j_\ell,a,b}| \delta_{c,d}^{e_\ell,f_\ell} + \delta_{a,b}^{e_\ell,f_\ell} |C_{j_\ell,c,d}| + \delta_{a,b}^{e_\ell,f_\ell} \delta_{c,d}^{e_\ell,f_\ell} \Big)}{|\mathcal{U}_i|} + O\Big(\frac{L_1(i)^3}{|\mathcal{U}_i|}\Big).
\end{aligned}$$

Since $Z_{e,f,a,b}(i) = Z_{a,b,e,f}(i)$ and $\sum_{j\in\mathfrak{C}_i} |C_{j,e,f}| = X_{e,f}(i)$ by (3.2) and (3.3), using $|\mathcal{U}_i| = U(i)$ it follows that the expected one-step changes of $Z_{a,b,c,d}(i)$ satisfy the *trend hypothesis*

$$\begin{aligned}
&\mathbb{E}(Z_{a,b,c,d}(i+1) - Z_{a,b,c,d}(i) \mid G_i) \\
&\quad = \frac{k(k-1) \sum_{(e,f)\in\mathcal{V}^*} \Big(Z_{a,b,e,f}(i) + \delta_{a,b}^{e,f} X_{e,f}(i) \Big) \sum_{(e,f)\in\mathcal{V}^*} \Big(Z_{e,f,c,d}(i) + \delta_{c,d}^{e,f} X_{e,f}(i) \Big)}{U(i)^2} \\
&\qquad + \frac{k \sum_{(e,f)\in\mathcal{V}^*} \Big(Z_{a,b,e,f}(i) \delta_{c,d}^{e,f} + \delta_{a,b}^{e,f} Z_{e,f,c,d}(i) + \delta_{a,b}^{e,f} \delta_{c,d}^{e,f} X_{e,f}(i) \Big)}{U(i)} + O\Big(\frac{L_1(i)^3}{U(i)}\Big).
\end{aligned} \tag{3.34}$$ eq:Z:change

The precise form of (3.34) not important for our purposes: what matters is that the expected one-step changes can be accurately estimated by a well-behaved function of the random variables $Z_{a,b,c,d}(i)$, $X_{e,f}(i)$ and $U(i)$. In particular, the differential equations (3.15)–(3.17) for $z_{a,b,c,d}(t)$ are again heuristically suggested by inserting the approximations $Z_{a,b,c,d}(i) \approx z_{a,b,c,d}(t)n$, $X_{e,f}(i) \approx x_{e,f}(t)n$ and $U(i) \approx u(t)n$ with $t = i/n$ into the expected one-step changes (3.34) and initial values (3.31), similarly to Section 3.3.1.

3.4 Deferred proof of main technical result

:proof

In this subsection we give the deferred proof of Theorem 3.1, which inductive proceeds roughly as follows. Assuming that (3.7)–(3.10) hold for step $i_{h-1} \approx t_{h-1}n$, we use the rough control result Theorem 2.1 to show that, with probability $1 - o(n^{-99})$, the technical exponential tail bound (3.10) again holds up to step $i_h \approx t_h n$. By Remark 3.2 this ensures that all components have size $O(\log n)$, so that the one-step changes (3.32) of the $Z_{a,b,c,d}(i)$ variables are at most $O((\log n)^2)$, i.e., remain fairly small. With this extra information about the subsequent evolution in hand, we then apply the differential equation method to show that, with probability $1 - o(n^{-99})$, the approximations (3.7)–(3.9) also hold up to step $i_h \approx t_h n$. Here the following simple lower bound on $U(i) = |\mathcal{U}_i|$ will be convenient: similarly to (3.24) we deterministically have

$$\frac{U(i)}{n} \geqslant \frac{\big(\sum_{v\in[n]} d_v^{(n)}/(kn) - t\big)k}{\Delta} \geqslant \frac{\big(T - t - o(1)\big)k}{\Delta} \qquad \text{for all } 0 \leqslant i \leqslant tn. \tag{3.35}$$

eq:U:lower

Proof of Theorem 3.1. We construct the claimed sequence by induction on h, using $\Psi_h(n) := 2h \cdot n^{-100}$ for concreteness. For the base case $h = 0$ we set $t_0 := 0$, $\beta_0 := 2$, $B_0 := 2$, and define $u(0)$, $x_{a,b}(0)$, $z_{a,b,c,d}(0)$ via the initial conditions (3.16). The inequalities in (P1) hold, since $z_{a,b,c,d}(0) \leqslant r_b \leqslant 1$ and $u(0) = \sum_{b\in[\Delta]} r_b \geqslant kT/\Delta$. Inspecting the initial values (3.25), (3.28), (3.31), there is a function $\xi_0(n) := \Delta\lambda_0(n) = o(1)$ which, for large n, deterministically satisfies (3.7)–(3.9) for $i = 0$ and the inequality in (P3). Inequality (3.10) also holds deterministically for $i = 0$, establishing the base case.

We now turn to the more interesting induction step, where $h \geqslant 1$. Let

$$i_{h-1} := \lfloor t_{h-1}n \rfloor, \tag{3.36}$$

ind:ih1

and define $\mathcal{E}_{h-1}$ as the event that $G_{i_{h-1}}$ satisfies (3.7)–(3.10) for $i = i_{h-1}$. By induction we have

$$\mathbb{P}(\neg\mathcal{E}_{h-1}) \leqslant \Psi_{h-1}(n). \tag{3.37}$$

eq:Pr:Eh

We henceforth condition on $G_{i_{h-1}}$, and assume that the event $\mathcal{E}_{h-1}$ holds. Noting that $s_u(t_{h-1})$ is inductively determined by (3.11) and (P1), gearing up to apply the rough control result Theorem 2.1 we define

$$\tau_h := \frac{1}{8k^2 \cdot \max\{s_u(t_{h-1}), 1\}}, \quad u_h := \frac{(T - t_{h-1})k}{2\Delta} \quad \text{and} \quad m_h := \lfloor \tau_h u_h n \rfloor. \tag{3.38}$$

def:tauh:uh:mh

Inequality (3.35) implies $|\mathcal{U}_{i_{h-1}}| \geqslant u_h n$ for large n, so that $m_h \leqslant \tau_h |\mathcal{U}_{i_{h-1}}|$. Similarly, the approximation (3.12) implies $S_u(i_{h-1}) \leqslant s_u(t_{h-1}) + 1$ for large n. By definition (3.38) of τ_h we infer $m_h \leqslant |\mathcal{U}_{i_{h-1}}|/(8k^2)$ and

$$k(k-1) \cdot m_h/|\mathcal{U}_{i_{h-1}}| \cdot S_u(i_{h-1}) \leqslant k(k-1) \cdot \tau_h \cdot \big(s_u(t_{h-1}) + 1\big) < 1/4. \tag{3.39}$$

eq:tail:ind:cond

Since $G_{i_{h-1}}$ satisfies (3.10) for $i = i_{h-1}$, by invoking inequality (2.1) from Theorem 2.1 with $i = i_{h-1}$, $m = m_h$, $\beta = \beta_{h-1}$, $B = B_{h-1}$, $\xi = \tau_h u_h/2$ and $\pi = 100$, there are constants $\alpha > 1$ and $A > 0$ such that

$$\mathbb{P}\big(\textstyle\sum_{j\in[n]} \alpha^j N_{\geqslant j}(G_{i_{h-1}+m_h}) \geqslant An \,\big|\, G_{i_{h-1}}\big) \leqslant n^{-100} \tag{3.40}$$

eq:tail:ind

for large n. Setting $\beta_h := \min\{\alpha, \beta_{h-1}\}$ and $B_h := \max\{A, B_{h-1}\}$, we now define the event

$$\mathcal{G}_h := \big\{\text{(3.10) holds for all } 0 \leqslant i \leqslant t_h n\big\}. \tag{3.41}$$

def:Gh

Note that $i_{h-1} + m_h \geqslant (t_{h-1} + \tau_h u_h/2)n = t_h n$ by (3.6). It follows by monotonicity of $N_{\geqslant j}(\cdot)$ that

$$\mathbb{P}(\neg\mathcal{G}_h) \leqslant \mathbb{P}(\neg\mathcal{E}_{h-1}) + \mathbb{P}\big(\text{(3.10) fails for } i \text{ with } i_{h-1} + m_h \leqslant i \leqslant t_h n, \text{ and } \mathcal{E}_{h-1} \text{ holds}\big) \leqslant \Psi_{h-1}(n) + n^{-100}. \tag{3.42}$$

eq:Pr:Gh

In preparation of the differential equation method, we observe that the event $\mathcal{G}_h$ implies

$$0 \leqslant Z_{a,b,c,d}(i)/n \leqslant S(i) = \sum_{j\geqslant 1} N_{\geqslant j}(G_i)/n \leqslant B_h \sum_{j\geqslant 1} \beta_h^{-j} \leqslant B_h/(\beta_h - 1) =: \Gamma_h \tag{3.43}$$

eq:Z:bound:Pi

for all $0 \leqslant i \leqslant t_h n$. With this in mind, we next define a domain $\mathcal{D}_h \subseteq \mathbb{R}^{2+|\mathcal{V}|+|\mathcal{V}|^2}$ which shall contain the functions $u(t)$, $x_{a,b}(t)$ and $z_{a,b,c,d}(t)$ for $t \in [0, t_h]$, while avoiding potential singularities of the functions $u(t)$ and $z_{a,b,c,d}(t)$. For concreteness fix $\varepsilon := 10^{-99}$ (any small constant suffices), and set

$$\mathcal{D}_h := \Big\{ \big(t,\, u,\, (x_{a,b})_{(a,b)\in\mathcal{V}},\, (z_{a,b,c,d})_{(a,b),(c,d)\in\mathcal{V}}\big) \;:\; t \in \big(-\varepsilon,\, t_h + (T-t_h)/2\big), \\ u \in \big(u_{h+1}/2,\, 1+\varepsilon\big),\; x_{a,b} \in \big(-\varepsilon,\, 1+\varepsilon\big),\; z_{a,b,c,d} \in \big(-\varepsilon,\, 2\Gamma_h\big) \Big\}. \tag{3.44}$$

def:Dh

Let us collect two consequences of the definitions of the domain $\mathcal{D}_h$ and the event $\mathcal{G}_h$. First, if we assume that $t \leqslant t_h + (T-t_h)/2$, then the deterministic inequality (3.35) implies $\min_{0\leqslant i\leqslant tn} U(i) \geqslant (T-t_h)n/(4\Delta) = \Omega(n)$ for large n, which ensures that the additive error terms in the expected one-step changes (3.27) and (3.30) of $X_{a,b}(i)$ and $U(i)$ are both $O(n^{-1})$. Second, if we assume that the graph G_i is consistent with the event $\mathcal{G}_h$, then Remark 3.2 implies $L_1(i) \leqslant C_h \log n$ when $i \leqslant t_h n$. For the variables $Z_{a,b,c,d}(i)$ with $i \leqslant t_h n$ this means that there is a constant $D'_h > 0$ such that (i) the additive error terms in the expected one-step changes (3.34) are bounded by $\delta := D'_h(\log n)^3/n$, and (ii) the maximum one-step changes (3.32) are at most $\beta := D'_h(\log n)^2$. Applying the differential equation method [53, 51] with the bounded domain $\mathcal{D} = \mathcal{D}_h$ to the variables $U(i)$, $X_{a,b}(i)$ and $Z_{a,b,c,d}(i)$, by the preparatory work of Section 3.3 it now is fairly routine to see that (a) the uniqueness property (P1) holds for $t \in [0, t_h]$, and (b) there is $\xi_h(n) = o(1)$ such that

$$\mathbb{P}\big((3.7)\text{–}(3.9) \text{ fail for some } 0 \leqslant i \leqslant t_h n, \text{ and } \mathcal{G}_h \text{ holds}\big) \leqslant n^{-\omega(1)}. \tag{3.45}$$

eq:Pr:dem:step

(For the interested reader, in the following short interlude we briefly expand on a few standard details regarding the application of the differential equation method, using [51, Theorem 2] for concreteness. The *trend hypothesis* corresponds to (3.13),(3.30), (3.14),(3.27), (3.15),(3.34) with additive error terms bounded by δ, the *boundedness hypothesis* corresponds to (3.26), (3.29), (3.32) with maximum one-step changes bounded by β, and the *initial condition* corresponds to (3.16), (3.28),(3.25),(3.31) with additive error terms bounded by $\lambda := \max\{\Delta\lambda_0(n), n^{-1/4}\} = o(1)$, say. To clarify: the improved estimates (i),(ii) may indeed be used for the variables $Z_{a,b,c,d}(i)$, since by [51, Lemma 9] we can abandon our argument for G_{i+1} as soon as G_i violates $\mathcal{G}_h$. The *Lipschitz hypothesis* is routinely verified: each of the derivatives in (3.13)–(3.15) corresponds to a function F that is a polynomial of degree at most two in the variables $(x_{a,b}/u)_{(a,b)\in\mathcal{V}}$ and $(z_{a,b,c,d}/u)_{(a,b),(c,d)\in\mathcal{V}}$, where u stays bounded away from 0 in the closure $\overline{\mathcal{D}}_h$ of $\mathcal{D}_h$, so $F : \overline{\mathcal{D}}_h \to \mathbb{R}$ has continuous derivatives in the compact set $\overline{\mathcal{D}}_h$, and is thus L-Lipschitz continuous in $\mathcal{D}_h$ for suitable $L = L(\overline{\mathcal{D}}_h)$. To ensure that the approximations (3.7)–(3.9) with $\xi_h(n) = O(\lambda) = o(1)$ extend to all $0 \leqslant i \leqslant t_h n$, it suffices to verify that the solutions $u(t)$, $x_{a,b}(t)$ and $z_{a,b,c,d}(t)$ to the system of differential equations (3.13)–(3.17) can only come $o(1)$ close to the boundary of $\mathcal{D}_h$ for $t \notin [0, t_h]$. This is straightforward, since otherwise we can easily get a contradiction to the fact that, for large n, the rescaled random variables are well approximated by the solutions of the differential equations. Indeed, $0 \leqslant X_{a,b}(i)/n \leqslant 1$ is trivial, (3.35) implies $u_{h+1} \leqslant U(i)/n \leqslant 1$ for $0 \leqslant i \leqslant t_h n$, and (3.43) implies $0 \leqslant Z_{a,b,c,d}(i)/n \leqslant \Gamma_h$ for $0 \leqslant i \leqslant t_h n$. Finally, the $n^{-\omega(1)}$ failure probability in (3.45) follows from $n\lambda^2/\beta^2 \gg \log n$ and $\lambda \gg \delta \gg n^{-1}$, see [51, Theorem 2 and Lemma 9].)

To sum up, combining (3.41)–(3.42) with (3.45) now completes (for large n) the proof of the induction step with $\Psi_h(n) := \Psi_{h-1}(n) + 2n^{-100} = 2h \cdot n^{-100}$, noting that $\xi_h(n) = o(1)$ implies the inequality in (P3). □

4 Random 2-process

rocess

In this section we prove Theorem 1.3 for the random 2-process $(G_i)_{i\geqslant 0} = (G^2_{n,i})_{i\geqslant 0}$ defined in Section 1; note that here we do *not* allow for loops or multiple edges. In particular, we shall analyze the size of the largest component in the final graph of the random 2-process using the following 2-*step argument*:

Step 1: Early evolution. We first consider the 2-process graph G_M after $M \approx n - o(n)$ steps, and show that it whp satisfies the following properties: (i) all components have size $o(n)$, and (ii) after removing an exceptional set W of $o(|\mathcal{U}_M|)$ vertices, all remaining active vertices in $\mathcal{U}_M \setminus W$ have degree 1 and are endpoints of paths with at least 3 vertices; see Section 4.1 and the event $\mathcal{T}$ in Section 4.2 for the details. For ease of exposition, we henceforth tacitly ignore the exceptional vertices in W. In particular, all relevant components[7]

[7]Here all cycles of G_M are irrelevant for our purposes, since they have size $o(n)$ and contain only inactive vertices.

of G_M containing active vertices are then simply paths with at least 3 vertices, whose set we denote by P. Clearly, from any such path $p \in P$ only its two endvertices can each participate in one more edge.

Step 2: Coupling with configuration model. The further evolution of the 2-process after step M then iteratively joins up endpoints of paths (also allowing paths to be closed to cycles), and in each step all such connections have the same probability[8] by definition of the 2-process. A coupling of the 2-process and the 2-regular configuration model [15, 52] with $m := |P|$ bins then becomes evident if we assign each path $p \in P$ to one bin with two points, so that path joinings in the 2-process correspond[9] to point pairings in the configuration model. To recover the component sizes in the 2-process using the configuration model, we need to take into account the sizes of the paths assigned to the bins, which are simply added up when distinct bins are paired up. By concentration of measure we expect that the resulting component sizes are close to their expected value (at least for large components), which in fact means that their size in the 2-process is approximately their configuration model size multiplied with a fixed 'stretching' factor, see (4.6). This stretching factor accounts for the fact that the configuration model has $m = o(n)$ bins, and it eventually cancels out after suitable rescaling of the sizes, see (4.5). In particular, it turns out that the size of the largest configuration model component rescaled by m is whp approximately equal to the size of the largest 2-process component rescaled by n, see (4.10). Using known results for the 2-regular configuration model, see Lemma 4.3, this then implies the desired distributional convergence result (1.3) of Theorem 1.3 for the largest component in the final graph of the 2-process; see Section 4.2 for a rigorous version of this heuristic argument (which also takes into account the exceptional set of vertices $W \subseteq \mathcal{U}_i$).

4.1 Early evolution and value of t_c

2pr:tc

In preparation of the proof of Theorem 1.3, we henceforth fix $0 < \delta \leqslant 1/70$ and define

$$M := \lceil n - n^{1-\delta} \rceil \quad \text{and} \quad r := \lfloor n^{1-\delta} \rfloor. \tag{4.1}$$

def:M:r

We first show that whp $L_1(G_M) = o(n)$, which in view of $M = n - o(n)$ easily implies $t_c = 1$.

:2proc

Lemma 4.1. *In the random 2-process, whp $L_1(G_M) \leqslant n^{1/2+5\delta}$.*

Proof. We first consider the event $\mathcal{A}$ that, during one of the first M steps, two components of size at least $s := n^{1/2+2\delta}$ join. Before completing step M, note that the number of active vertices is always at least $n - (M-1) \geqslant n^{1-\delta} =: a$ (analogous to (3.35) in Section 3.4), and that the number of components of size at least s is always at most n/s. Since any component contains at most two active vertices, by taking a union bound over all M possible joining steps it follows that

$$\mathbb{P}(\mathcal{A}) \leqslant M \cdot O\big((n/s)^2 \cdot a^{-2}\big) = O(n^{-2\delta}).$$

We next consider the event $\mathcal{B}_i$ that, during one of the first M steps, the component containing vertex i joins at least $z := \lceil n^{2\delta} \rceil$ times another component. With similar reasoning as for the event $\mathcal{A}$ above, by taking a union bound over the first z joining steps it is routine to see that

$$\mathbb{P}(\mathcal{B}_i) \leqslant \tbinom{M}{z} \cdot \big[O\big(a^{-1}\big)\big]^z \leqslant \big[O(M/(az)\big]^z \leqslant e^{-z} = o(n^{-1-2\delta}).$$

Hence, using a union bound argument, whp none of the events $\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n$ occur. In that case, any component initially starts with one vertex i and during the first M steps then sequentially grows by at most z component joinings, which each bring in at most s new vertices. Thus $L_1(G_M) \leqslant 1 + zs = o(n^{1/2+5\delta})$. □

:2proc

Corollary 4.2. *The critical point of the random 2-process satisfies $t_c = 1$.*

Proof. Note that $t_c \leqslant T = 1$ follows by applying Theorem 1.6 to the 2-process, with $k = \Delta = 2$, $\mathbf{r} = (0, 0, 2)$ and $\mathbf{d}_n = (2, \dots, 2)$. Now, aiming at a contradiction, suppose that $t_c < T$. Pick $t_c < t < T$. Applying Lemma 4.1 and Theorem 1.6 implies that whp $cn \leqslant L_1(G_{\lfloor tn \rfloor}) \leqslant n^{1/2+5\delta} < n^{3/4}$, which for large n yields the desired contradiction. □

Hence Remark 1.4 for the random 2-process is a direct consequence of Theorem 1.6.

[8] This uniformity would fail if 2-vertex paths were in P, since pairing their two endpoints yields a forbidden multiple edge.
[9] This correspondence would fail if isolated vertices were in P, since pairing their two bin points yields a forbidden loop.

4.2 Final graph: size of the largest component

g:size

To study the final graph of the 2-process, we now consider the further evolution from step M onwards; here $M = \lceil n - n^{1-\delta} \rceil$ and $r = \lfloor n^{1-\delta} \rfloor$ as well as $0 < \delta \leqslant 1/70$ are defined as in (4.1) above.

Proof of Theorem 1.3. We start with the auxiliary claim that G_M satisfies, whp, the following event $\mathcal{T}$:

i:P:2 i:P:1

(i) The largest component has size at most $n^{1/2+5\delta} = o(r)$.

(ii) The number X_i of vertices of degree i satisfies $X_1 \sim 2r$ and $X_0 = o(r)$. Additionally, the number Y of components of size 2 satisfies $Y = o(r)$.

Indeed, (i) is implied by Lemma 4.1, and (ii) follows from [48, Theorem 4 and (2.23)] (and requires $\delta < 1/4$), establishing the claim. We henceforth condition on G_M, and assume that $\mathcal{T}$ holds. Noting that X_1 is even, the components of the graph G_M must consist of X_0 isolated vertices, Y paths with 2 vertices, and

$$m := X_1/2 - Y \sim r \tag{4.2}$$

def:m

paths with at least 3 vertices, and an additional set C of cycles (if there are any). For later reference, we write P for the set of $m = |P|$ paths with at least 3 vertices, and W for the set of $|W| = X_0 + 2Y = o(r)$ vertices in components of size at most 2. We also denote by $\mathcal{G}$ the random 2-process applied from this point onwards.

To analyze how the subsequent steps of $\mathcal{G}$ affects these paths and components, we shall below employ the standard configuration model [15, 52] for a random 2-regular graph on m vertices. This has m bins with 2 points in each, and a random pairing (matching) of the points is chosen. The bins are collapsed into vertices, the pairs become edges, and a multigraph (possibly with loops and multiple edges) results. We consider the configuration model process $\mathcal{R} = (R_0, \dots, R_m)$ which arises by choosing the random pairs of points sequentially. Here R_i is marginally distributed as a uniformly random choice of a set of i pairs of the $2m$ points in the model.

The key point is that we can define a coupling of (a sub-process of) the 2-process $\mathcal{G}$ with the configuration model process $\mathcal{R}$ as follows. Each path $p \in P$ represents a bin in $\mathcal{R}$, which initially contains two points that are labeled by the two endpoints of p. Sequentially considering the steps of the 2-process $\mathcal{G}$, suppose that v_1v_2 is added in the current step. If both v_1 and v_2 currently appear as labels in $\mathcal{R}$, then we add the corresponding pair v_1v_2 in the configuration process $\mathcal{R}$. Otherwise no step is taken in $\mathcal{R}$, but for coupling purposes we update the labels as follows: if only v_1 appears as a label in $\mathcal{R}$, then v_2 is one endpoint of a path p' in $\mathcal{G}$ whose vertices are all in W, and we replace the label v_1 by the other endpoint of p' (to clarify: this endpoint equals v_2 when p' consists only of one vertex); we proceed analogously when only v_2 appears as a label in $\mathcal{R}$, with the roles of v_1 and v_2 interchanged. Note that in $\mathcal{R}$ is each so-far unpaired pair with labels vw corresponds to an edge vw that can be added to $\mathcal{G}$ (since both v, w currently have degree one, and are not yet connected by an edge). Under this coupling, it thus follows that in each step of $\mathcal{R} = (R_1, \dots, R_m)$ all pairs of currently unpaired vertices are indeed equally likely to be joined up, as desired.

Consider the auxiliary graph R'_m obtained by replacing each vertex in the multigraph resulting from R_m by its corresponding path $p \in P$, in the obvious way so that the maximum degree is 2. The components of R'_m are just cycles, whose size (number of vertices) equals the total size of those paths $p \in P$ it contains. Note that the component structure at the end of the 2-process $\mathcal{G}$ is obtained from the auxiliary graph R'_m by (a) inserting the vertices of W into cycles of R'_m or into separate other cycles or paths, and then (b) adding the set C of cycles from G_M. Since $\mathcal{T}$ holds, the insertions and additions in (a) only involve at most $o(r)$ new vertices, and the cycles added in (b) each have size at most $o(r)$. It follows that

$$\big|L_1(G_n) - L_1(R'_m)\big| = o(r), \tag{4.3}$$

def:L1diff

where we temporarily write G_n for the graph at the end of the 2-process to avoid clutter (this slight abuse of notation will later be justified by the fact that this final graph has whp n edges).

The next step is to relate the auxiliary graph R'_m to the 2-regular configuration model R_m. Here the key observation is that, equivalently to the process described above, we can obtain R'_m by first running the process $\mathcal{R}$ to determine R_m, and only after that assigning the paths in P randomly to the vertices of R_m. Writing $s_1, \dots, s_m$ for the sizes (number of vertices) of the paths in P, note that $\mathcal{T}$ implies

$$w := \sum_{j \in [m]} s_j \in \big[n - r\,, n\big] \quad \text{and} \quad S := \max_{j \in [m]} s_j \leqslant n^{1/2+5\delta}. \tag{4.4}$$

eq:sumsj:S

Consider any cycle Q of R_m of length $q \geqslant 1$. After replacement of vertices by paths, the resulting cycle size in R'_m is equal to the sum of q numbers sampled uniformly at random from the sequence $(s_1, \dots, s_m)$, without replacement. Letting Z_i denote the i-th number chosen, the resulting cycle size is thus $\sum_{i\in[q]} Z_i =: X_Q$. We now claim that the size $L_1(R'_m) = \max X_Q$ of the largest component of R'_m whp satisfies

$$\big|L_1(R'_m)/n - L_1(R_m)/m\big| \leqslant 4n^{-\delta}. \tag{4.5}$$ eq:CycRm

To see this, let $q_0 := \sqrt{m^{3/2}n^{8\delta}}$. First define $\mathcal{L}$ as the event that, for all $q \geqslant q_0$, every cycle Q of R_m with size q satisfies $X_Q = (1 \pm n^{-\delta})qw/m$. The expectation and variance of X_Q are known (see, e.g., Section 6 in [27] for the exact relation between sampling with and without replacement) to satisfy

$$\mathbb{E}(X_Q \mid R_m) = q \cdot \frac{w}{m} \qquad \text{and} \qquad \mathrm{Var}(X_Q \mid R_m) \leqslant S \cdot \mathbb{E}(X_Q \mid R_m) = \frac{Sqw}{m}. \tag{4.6}$$ eq:exp:XQ:var

Note that R_m contains at most m/q_0 cycles of size at least q_0. Taking a union bound argument over all cycles, using Chebyshev's inequality together with (4.4) and $m \leqslant n$ it follows that

$$\mathbb{P}(\neg\mathcal{L}) = \mathbb{E}\mathbb{P}(\neg\mathcal{L} \mid R_m) \leqslant \frac{m}{q_0} \cdot \max_{q\geqslant q_0} \frac{Sm}{n^{-2\delta}qw} = \frac{n^{2\delta}Sm^2}{q_0^2 w} = O(n^{-\delta}). \tag{4.7}$$ eq:Cheb:XQ:gamma

Next define $\mathcal{S}$ as the event that, for all $q \leqslant q_0$, every cycle Q of R_m with size q satisfies $X_Q \leqslant n^{1-\delta}$. Known variants of the Azuma–Hoeffding and Chernoff bounds (see, e.g., Theorem 2 and the remark below Theorem 4 in [27] for upper tail inequalities that apply to sampling without replacement) yield for all $x > 0$ that the upper tail $\mathbb{P}(X_Q \geqslant qw/m + x|R_m)$ is at most $\exp(-2x^2/(qS^2))$. Note that $n^{1-\delta} \gg q_0 m/w$. Taking a union bound similar to (4.7) above, using $q_0 < n^{3/4+4\delta}$ and $\delta \leqslant 1/70$ it follows that

$$\mathbb{P}(\neg\mathcal{S}) \leqslant n \cdot \max_{q\leqslant q_0} \exp\Big(-\Omega\big(n^{1-12\delta}/q\big)\Big) \leqslant n \cdot \exp\Big(-\Omega\big(n^{1/4-16\delta}\big)\Big) = o(n^{-\delta}). \tag{4.8}$$ eq:Chern:XQ:gamma

Noting $L_1(R_m)w/m \leqslant w \leqslant n$ and $q_0 w/m = o(n^{1-\delta})$, the event $\mathcal{L}\cap\mathcal{S}$ implies

$$\Big|L_1(R'_m) - L_1(R_m)\frac{w}{m}\Big| \leqslant n^{-\delta} \cdot L_1(R_m)\frac{w}{m} + 2n^{1-\delta} \leqslant 3n^{1-\delta},$$

which together with $L_1(R_m)|w-n|/m \leqslant r \leqslant n^{1-\delta}$ establishes that (4.5) holds whp, as claimed.

It remains to analyze the size $L_1(R_m)$ of the largest component in the 2-regular configuration model R_m. Results of type (4.9) are known, but often proved in a more technical setting (such as Table 2.2, Lemma 5.7, page 110 and Theorem 6.8 in [5]); we thus include an elementary proof of Lemma 4.3 in Appendix C.3.

m:L1Rm **Lemma 4.3.** *There exists a continuously decreasing function $F : (0,1] \to [0,1]$ such that*

$$\lim_{m\to\infty} \mathbb{P}(L_1(R_m)/m > c) \;=\; F(c) \qquad \textit{for any } c \in (0,1], \tag{4.9}$$ def:LF

with $F(c) \in (0,1)$ for $c \in (0,1)$, $F(c) = \log\big(\sqrt{c^{-1}} + \sqrt{c^{-1}-1}\big)$ for $c \in (1/2, 1]$, and $F(\varepsilon) \to 1$ as $\varepsilon \to 0$.

To sum up, since the event $\mathcal{T}$ holds whp and the 2-process ends whp after n steps by [42] (justifying our slight abuse of notation in (4.3) above), by combining (4.3) and (4.5) with $r \leqslant n^{1-\delta}$ it follows that, whp,

$$\big|L_1(G_n)/n - L_1(R_m)/m\big| \leqslant o(r/n) + 4n^{-\delta} \leqslant 5n^{-\delta}. \tag{4.10}$$ def:L1GnRm

Given $c \in (0,1)$, we fix $\varepsilon > 0$ small enough such that $[c-\varepsilon, c+\varepsilon] \subset (0,1)$, and infer

$$\mathbb{P}\big(L_1(R_m)/m > c+\varepsilon\big) - o(1) \;\leqslant\; \mathbb{P}\big(L_1(G_n)/n > c\big) \;\leqslant\; \mathbb{P}\big(L_1(R_m)/m > c-\varepsilon\big) + o(1). \tag{4.11}$$ def:LF:G

Inserting (4.9) into (4.11), now (1.3) follows by first sending $n \to \infty$ (and thus $m \sim r = \lfloor n^{1-\delta} \rfloor \to \infty$) and afterwards sending $\varepsilon \searrow 0$, noting that $F(c \pm \varepsilon) \to F(c) \in (0,1)$. This completes the proof of Theorem 1.3 since trivially $\mathbb{P}\big(L_1(G_n)/n > 1\big) = 0 = F(1)$. □

The arguments of this section extend to the sizes $L_1(n), \dots, L_r(n)$ of the largest $r = O(1)$ components in the final graph G_n of the random 2-process. Indeed, noting the second proof of $F(c) < 1$ in the proof of Lemma 4.3 given in Appendix C.3, by combing the above concentration event $\mathcal{L}\cap\mathcal{S}$ with a minor variant of the approximation inequalities (4.3) and (4.10) it is not hard to extend (1.3) and deduce that

$$\lim_{n\to\infty} \mathbb{P}\big(L_1(n)/n > c_1,\; L_2(n)/n > c_2,\; \dots,\; L_r(n)/n > c_r\big) \;=\; F(c_1, \dots, c_r) \tag{4.12}$$ eq:L1sup:multiple

for a certain function F that is strictly between 0 and 1 provided $0 < \sum_{i\in[r]} c_i < 1$.

5 Numerical estimates of critical point

apx:tc

In this section we demonstrate that our methods from Sections 2–3 also give (without much effort) further information about the critical point t_c. Indeed, the following lemma shows that if we are able to numerically solve the differential equations (3.13)–(3.17) for the idealized active susceptibility $s_u(t)$ with explicit error bounds, then we can use this to (i) prove that $t_c < T$ holds, and (ii) estimate t_c up to arbitrary precision.

umeric

Lemma 5.1. *Suppose that the assumptions of Theorem 1.6 hold. Let $T := \sum_{j\in[\Delta]} jr_j/k$. Define $s_u(t)$ as in (3.11), and t_c as in (3.18). Let $\alpha := 1/(32k\Delta)$ and $\beta := 64/k$. If $t_c < T$, then $\lim_{t\nearrow t_c} s_u(t) = \infty$. If $t \in [0,T)$ satisfies $\max_{x\in[0,t]} s_u(x) < \infty$ and $s_u(t) \geqslant 130$, then $t_c < T$ and*

$$t_c \in \left(t + \frac{\alpha(T-t)}{s_u(t)},\ t + \frac{\beta(T-t)}{s_u(t)-2}\right). \tag{5.1}$$

eq:sut:tc

umeric

Remark 5.2. *The proof shows more generally that, for $t \in [0,t_c)$, we have*

$$\frac{\alpha\Delta u(t)}{k(t_c-t)} < s_u(t) < \max\Big\{\frac{\beta u(t)}{k(t_c-t)},\ 128\Big\} + 2, \tag{5.2}$$

eq:sut:bound

$$k(T-t)/\Delta \leqslant u(t) \leqslant k(T-t). \tag{5.3}$$

eq:ubounds

The proof is based on the idea that if $s_u(t)$ is too small or big for $t \in [0,t_c)$, then we can reach a contradiction between the whp approximation $S_u(tn) \approx s_u(t)$ and the phase transition location t_c. To this end we again consider the multigraph variant $(G_i)_{i\geqslant 0} = (G_{n,i}^{k,\mathbf{d}_n})_{i\geqslant 0}$, as in Sections 2–3.

Proof (sketch). Note that all estimates claimed by Lemma 5.1 follow from Remark 5.2, so it suffices to prove inequalities (5.2)–(5.3). Noting that $|\mathcal{U}_i|\Delta \geqslant \sum_{v\in[n]}(d_v^{(n)} - \deg_{G_i}(v)) \geqslant |\mathcal{U}_i|$, by proceeding along the lines of (3.35) and the proof of Corollary 3.4, using the whp approximation (3.7) we infer (5.3).

Aiming at a contradiction, suppose that $s_u(t) \leqslant \alpha\Delta u(t)/[k(t_c-t)] =: \pi$ for some $t \in [0,t_c)$, where $s_u(t) \geqslant 1$ implies $\pi \geqslant 1$. Using Theorem 3.1 we condition on the whp event that G_i satisfies (3.7)–(3.10) for $i := \lceil tn\rceil$. Let $\tau := 1/(8k^2\pi)$ and $m := \lfloor \tau|\mathcal{U}_i|\rfloor$. Noting $m \leqslant |\mathcal{U}_i|/(8k^2)$ and $k(k-1)\cdot m/|\mathcal{U}_i| \cdot S_u(i) \leqslant k^2\cdot\tau\cdot 2s_u(t) = 1/4$, inequality (2.1) from Theorem 2.1 implies that whp $L_1(G_{i+m}) \leqslant C\log n$ for some $C > 0$. But, noting $i+m \geqslant [t+\tau u(t)/2]n = [t_c + (t_c-t)]n$, the multigraph variant of Theorem 1.6 (as proved in Section 3.2.2) also implies that whp $L_1(G_{i+m}) \geqslant cn$, which for large n yields the desired contradiction.

Aiming at a contradiction, suppose that $s_u(t) \geqslant \max\{\beta u(t)/[k(t_c-t)], 128\} + 2 =: \lambda + 2$ for some $t \in [0,t_c)$. Using Theorem 3.1 we condition on the whp event that G_i satisfies (3.7)–(3.10) for $i := \lfloor tn\rfloor$. Let $\tau := 16/(k^2\lambda)$ and $m := \lfloor \tau|\mathcal{U}_i|\rfloor$. Noting $m \leqslant |\mathcal{U}_i|/(8k^2)$ and $k(k-1)\cdot m/|\mathcal{U}_i| \cdot (S_u(i)-1) \geqslant k^2/2\cdot\tau/2\cdot\lambda = 4$, inequality (2.2) from Theorem 2.1 implies that whp $L_1(G_{i+m}) \leqslant cn$. But, noting $i+m \leqslant [t+2\tau u(t)]n \leqslant [t_c - (t_c-t)/2]n$, the multigraph variant of Theorem 1.6 (as proved in Section 3.2.2) also implies that whp $L_1(G_{i+m}) \leqslant C\log n$, which for large n yields the desired contradiction. □

The above arguments illustrate one conceptual difference to the statistical physics approach [10] for the random d-process, which for each $d \geqslant 3$ deduces[10] existence of the critical $t_c = t_c(d)$ by numerically finding the point beyond which a certain integral equation (involving the solution to a nonlinear second order differential equation) starts having multiple solutions. Indeed, in our approach we analytically prove the existence of the critical point t_c, and merely use numerical methods to estimate the concrete value of t_c (if desired).

6 Open problems

c:open

In this paper we initiated the study of the phase transition in the random d-process and its degree restricted hypergraph generalizations, and the natural next step is to study the finer details of the phase transition. With the goal of making the proof techniques in the area more robust (i.e., to handle non-trivial dependencies

[10] For the random d-process with fixed $d \geqslant 1$, their approach uses (among other things) the scaling assumption [10, equation (32)] for the fraction of vertices in the giant component. This plausible assumption is somewhat non-trivial here, since it fails for the 2-process due to the potential presence of multiple giant components, see (4.12).

between edges), here the following open problems for the random k-uniform $\mathbf{d}_n$-process seem interesting for further work. Below we tacitly assume that the general assumptions of Theorem 1.6 hold, but we stress that our main interest is really the graph case $k = 2$, in particular the random d-process (whenever relevant).

• **Critical point:** Determine the correct condition for $t_c < T$, i.e., when the giant component emerges significantly before the end of the process. Perhaps optimistically, we conjecture that $t_c < T$ is implied[11] by $(k-1)\sum_{j\in[\Delta]}(j-1)jr_j/\sum_{j\in[\Delta]} jr_j > 1$, i.e., the hypergraph generalization of the (branching process based) Molloy-Reed giant component criterion for uniform random graphs with specified degrees. Of course, it would also be interesting to get more explicit analytical knowledge of t_c, and here it might be insightful to first prove that $t_c = t_c(d) > 1/2$ for the d-process with $d \geqslant 3$.

• **Maximum degree:** Remove or weaken the technical condition $\Delta = O(1)$ from Theorem 1.6.

• **Second largest component:** Show that the second largest supercritical component has logarithmic size, i.e., that for any $t \in (t_c, T)$ there is $C = c(t, k, \Delta, \mathbf{r}) > 0$ such that whp $L_2(\lfloor tn \rfloor) \leqslant C \log n$.

• **Giant component:** (i) Show that the giant component has a scaling limit, i.e., that there is a function $\rho = \rho^{k,\Delta,\mathbf{r}} : (t_c, T) \to (0, 1]$ such that, for any $t \in (t_c, T)$ and $\delta > 0$, we have $\mathbb{P}(|L_1(\lfloor tn \rfloor)/n - \rho(t)| \geqslant \delta) \to 0$ as $n \to \infty$. (ii) Show the phase transition is 'second order', i.e., that $\rho(t_c + \varepsilon) \sim c\varepsilon$ as $\varepsilon \searrow 0$ for some constant $c = c(k, \Delta, \mathbf{r}) > 0$. (iii) Show that the giant component satisfies a central limit theorem, i.e., that after suitable rescaling $L_1(\lfloor tn \rfloor)$ is asymptotically normal for any $t \in (t_c, T)$; cf. [45].

• **Nearly 2-regular case:** Determine the behavior of the largest component in the final graph when $k = 2$ and $r_2 = 1$. To clarify: the 2-process corresponds to the case when all n degree-restrictions $d_v^{(n)}$ are 2, and we are asking about the behavior when only $n - o(n)$ degree restrictions $d_v^{(n)}$ are 2. More generally, it would be interesting to get a better understanding[12] of the final hypergraph behavior when $t_c = T$.

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A Critical point of random d-process when $d \to \infty$

px:ERR

In this appendix we study the location $t_c = t_c(d)$ of the phase transition in the d-process for large d.

m:crit

Theorem A.1. *The critical point $t_c = t_c(d)$ of the random d-process satisfies $t_c(d) \to 1/2$ as $d \to \infty$, and $t_c(d) = 1/2$ when $d = d(n) \to \infty$ as $n \to \infty$.*

The proof is based on a comparison of the random d-process $(G^d_{n,i})_{i\geqslant 0}$ with the classical Erdős–Rényi random graph process $(G_{n,i})_{i\geqslant 0}$. The crux is that the Erdős–Rényi giant component (i) emerges after roughly $n/2$ steps, and (ii) is robust with respect to the deletion of a few edges. Furthermore, for large d, most vertices of the Erdős–Rényi random graph $G_{n,n}$ have degree less than d. By viewing $(G^d_{n,i})_{i\geqslant 0}$ as a process where we ignore certain edges from $(G_{n,i})_{i\geqslant 0}$, this makes it plausible that $t_c(d) \to 1/2$ as $d \to \infty$.

Proof. We consider $(e_j)_{j\geqslant 1}$, where each edge e_{j+1} is chosen uniformly at random from $\binom{[n]}{2} \setminus \{e_1 \ldots, e_j\}$. Clearly, $\{e_1, \ldots, e_i\}$ gives the uniform random graph $G_{n,i}$. Furthermore, we obtain the evolution of the d-process by sequentially traversing the $(e_j)_{j\geqslant 1}$, only adding those edges which do not create a vertex of degree larger than d. Since e_j is certainly added by the d-process if both its endvertices appear in less than d edges in $\{e_1, \ldots, e_{j-1}\}$, this yields a natural coupling with the property that $G^d_{n,\max\{0,i-X_{n,d}\}} \subseteq G_{n,i}$ for all $0 \leqslant i \leqslant n$, with $X_{n,d} := \sum_{k\geqslant d} kD_k$ and $D_{n,k}$ denoting the number of vertices in $G_{n,n}$ with degree k (see [18, 37, 49] for related arguments). Writing $p = n/\binom{n}{2} \sim 2/n$ and noting $\mathbb{E}D_{n,k} \leqslant n \cdot \binom{n}{k} p^k \leqslant n(6/k)^k$, a standard first moment argument shows that whp $\max_{k\geqslant \log n} D_{n,k} = 0$. Together with a routine application of the bounded differences inequality for uniform random graphs (see, e.g., [35, Theorem 7.4 and Example 7.3] or the discussion below [50, Theorem 1.9]), there are constants $\beta \in (0,1)$ and $B \in [1,\infty)$ such that, whp,

$$X_{n,d} = \sum_{d\leqslant k<\log n} kD_{n,k} \leqslant \sum_{d\leqslant k<\log n} k\big(\mathbb{E}D_{n,k} + n^{2/3}\big) \leqslant \big\lceil B\max\big\{\beta^d,\ n^{-1/4}\big\}n\big\rceil =: I_{n,d}.$$

Putting things together, the described natural coupling whp satisfies the following inclusion:

$$G^d_{n,i-I_{n,d}} \subseteq G_{n,i} \qquad \text{for all } I_{n,d} \leqslant i \leqslant n. \tag{A.1}$$

eq:thm:crit:couple

We now compare the d-process with the Erdős–Rényi process. In the subcritical case, for any $\varepsilon \in (0,1/2)$ we have $\varepsilon n > \varepsilon n/2 + I_{n,d}$ for $d \geqslant d_1(\varepsilon)$ and $n \geqslant n_1(\varepsilon)$, so using (A.1) it follows that, whp,

$$L_1(G^d_{n,(1/2-\varepsilon)n}) \leqslant L_1(G^d_{n,(1/2-\varepsilon/2)n-I_{n,d}}) \leqslant L_1(G_{n,(1/2-\varepsilon/2)n}) \leqslant C_\varepsilon \log n, \tag{A.2}$$

eq:thm:crit:lower

where the last inequality is well-known, see [31, Theorem 5.4]. For the supercritical case we define $L_1^x(G) := \min L_1(G')$, where the minimum is taken over all graphs G' that differ from G in at most x edges. It is known[13]

[13] Writing $m := (1/2+\varepsilon)n$ and $p := m/\binom{n}{2} \sim (1+2\varepsilon)/n$, for the binomial random graph $G_{n,p}$ whp $L_1^{\kappa n}(G_{n,p}) \geqslant c_\varepsilon n$ by [17, Theorem 3.9], which carries over to the uniform random graph $G_{n,m}$ by monotonicity [31, Corollary 1.16(i)].

that whp $L_1^{\kappa n}(G_{n,(1/2+\varepsilon)n}) \geqslant c_\varepsilon n$ for suitable $\kappa = \kappa(\varepsilon) > 0$. For any $\varepsilon \in (0, 1/2)$ we have $I_{n,d} < \min\{\kappa n, n/2\}$ for $d \geqslant d_2(\varepsilon)$ and $n \geqslant n_2(\varepsilon)$, so using the coupling from (A.1) it follows that, whp,

$$L_1(G^d_{n,(1/2+\varepsilon)n}) \geqslant L_1(G^d_{n,(1/2+\varepsilon)n-I_{n,d}}) \geqslant L_1^{\kappa n}(G_{n,(1/2+\varepsilon)n}) \geqslant c_\varepsilon n. \tag{A.3}$$

eq:thm:crit:upper

To sum up, since $t_c(d)$ exists for fixed $d \geqslant 3$ by Theorem 1.1, the whp estimates (A.2)–(A.3) imply that for any $\varepsilon > 0$ we have $|t_c(d) - 1/2| \leqslant \varepsilon$ for $d \geqslant d_0(\varepsilon)$, i.e., that $t_c(d) \to 1/2$ as $d \to \infty$. The same reasoning also establishes that $t_c(d) = 1/2$ when $d = d(n) \to \infty$ as $n \to \infty$, without presupposing the existence of $t_c(d)$. □

We remark that when $d = d(n) \to \infty$ as $n \to \infty$, then the above proof also shows that, for any $t = O(1)$, the size of the largest component in the d-process whp satisfies (1.2) with $t_c = 1/2$.

B Transferring results from multigraph variant

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In this appendix we show that whp results for the k-uniform $\mathbf{d}_n$-process routinely follow from whp results for its multigraph variant. Indeed, given $t \in [0, T)$, note that if an event $\mathcal{E}$ fails with probability at most π in the multigraph variant $(G^{k,\mathbf{d}_n}_{n,i})_{0\leqslant i\leqslant tn}$, then by inequality (B.1) this event $\mathcal{E}$ fails with probability at most $C \cdot \pi = O(\pi)$ in the original process $(H^{k,\mathbf{d}_n}_{n,i})_{0\leqslant i\leqslant tn}$. This justifies Remark 1.8 from Section 1.4.1, since in Theorem 1.6 we only consider fixed $t \in [0, T)$ and events that fail with probability at most $o(n^{-99})$.

ansfer

Lemma B.1. *Suppose that $k \geqslant 2$, $\Delta \geqslant 1$, $\mathbf{r} = (r_0, \dots, r_\Delta) \in [0,1]^{\Delta+1}$ and $\mathbf{d}_n = \big(d_1^{(n)}, \dots, d_n^{(n)}\big) \in \{0, \dots, \Delta\}^n$ satisfy the assumptions of Theorem 1.6. Set $T := \sum_{j\in[\Delta]} jr_j/k$. Then for every $t \in [0, T)$ there exists $C = C(t, k, \Delta, T) > 0$ such that, for n large enough, we have*

$$\mathbb{P}\big(\big(H^{k,\mathbf{d}_n}_{n,i}\big)_{0\leqslant i\leqslant tn} \in \mathcal{S}_{n,tn}\big) \;\leqslant\; C \cdot \mathbb{P}\big(\big(G^{k,\mathbf{d}_n}_{n,i}\big)_{0\leqslant i\leqslant tn} \in \mathcal{S}_{n,tn}\big) \tag{B.1}$$

eq:transfer

for any set $\mathcal{S}_{n,tn}$ of hypergraph sequences $(G_i)_{0\leqslant i\leqslant tn}$ with vertex set $[n]$.

The proof is based on a standard step-by-step comparison argument (similar to [25, 38, 29]).

Proof. We fix a hypergraph sequence $(G_i)_{0\leqslant i\leqslant tn} \in \mathcal{S}_{n,tn}$ that can be attained by $\big(H^{k,\mathbf{d}_n}_{n,i}\big)_{0\leqslant i\leqslant tn}$. Comparing the probabilities with which the next edge is added in each of the two process, we have

$$\mathbb{P}\Big(H^{k,\mathbf{d}_n}_{n,i+1} = G_{i+1} \,\Big|\, \bigcap_{0\leqslant j\leqslant i} \big\{H^{k,\mathbf{d}_n}_{n,j} = G_j\big\}\Big) \;=\; \frac{|\mathcal{U}_i|^k/k!}{|\mathcal{Q}_i|} \cdot \mathbb{P}\Big(G^{k,\mathbf{d}_n}_{n,i+1} = G_{i+1} \,\Big|\, \bigcap_{0\leqslant j\leqslant i} \big\{G^{k,\mathbf{d}_n}_{n,j} = G_j\big\}\Big), \tag{B.2}$$

eq:transfer:ind

where $\mathcal{Q}_i$ denotes the set of edges that can be added to $H^{k,\mathbf{d}_n}_{n,i}$, and $\mathcal{U}_i$ denotes the set of active vertices in G_i. Similar to (3.35), we deterministically have $\min_{0\leqslant i\leqslant tn} |\mathcal{U}_i| \geqslant (T - t)n/\Delta > 2(k^2 + \Delta k!)$ for n large enough. Noting $\big|\binom{\mathcal{U}_i}{k} \cap E(G_i)\big| \leqslant |\mathcal{U}_i| \cdot \Delta$ and $\binom{|\mathcal{U}_i|}{k} \geqslant (1 - k/|\mathcal{U}_i|)^k \cdot |\mathcal{U}_i|^k/k!$, for $0 \leqslant i \leqslant tn$ we thus infer

$$\frac{|\mathcal{U}_i|^k/k!}{|\mathcal{Q}_i|} \leqslant \frac{|\mathcal{U}_i|^k/k!}{\binom{|\mathcal{U}_i|}{k} - |\mathcal{U}_i|\Delta} \leqslant \frac{1}{1 - (k^2 + \Delta k!)/|\mathcal{U}_i|} \leqslant \exp\Big(\frac{2(k^2 + \Delta k!)\Delta}{(T - t)n}\Big).$$

Since initially $\mathbb{P}(H^{k,\mathbf{d}_n}_{n,0} = G_0) = 1 = \mathbb{P}(G^{k,\mathbf{d}_n}_{n,0} = G_0)$, by multiplying (B.2) it follows that

$$\mathbb{P}\Big(\bigcap_{0\leqslant i\leqslant tn} \big\{H^{k,\mathbf{d}_n}_{n,i} = G_i\big\}\Big) \;\leqslant\; \exp\Big(\frac{2t(k^2 + \Delta k!)\Delta}{(T - t)}\Big) \cdot \mathbb{P}\Big(\bigcap_{0\leqslant i\leqslant tn} \big\{G^{k,\mathbf{d}_n}_{n,i} = G_i\big\}\Big). \tag{B.3}$$

eq:transfer:prod

This establishes (B.1) by summing (B.3) over all attainable sequences $(G_i)_{0\leqslant i\leqslant tn} \in \mathcal{S}_{n,tn}$. □

C Deferred proofs

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C.1 Lemma 2.3: subcritical case (adding random edges)

:subcr

In this appendix we prove the subcritical case of Lemma 2.3 from Section 2.2. For technical reasons we consider a Poissonized variant of $H_\ell = F + \mathcal{E}_{\ell,W}$, and introduce the multiset $\mathcal{E}^*_{\ell,W}$ where each of the $|W|^k$ tuples $(v_1,\dots,v_k) \in W^k$ arrives according to independent Poisson processes with mean $\psi := \ell/|W|^k$. We study

$$H^*_\ell := F + \mathcal{E}^*_{\ell,W} = \big(V(F),\, E(F) \cup \mathcal{E}^*_{\ell,W}\big),$$

where we tacitly map tuples $(v_1,\dots,v_k)$ to hyperedges $\{v_1,\dots,v_k\}$. Our upcoming analysis exploits standard splitting properties of Poisson processes, which imply that we may generate $\mathcal{E}^*_{\ell,W}$ via the following more tractable two-stage process: we first determine $x := |\mathcal{E}^*_{\ell,W}| \sim \mathrm{Po}(\ell)$, and then set $\mathcal{E}^*_{\ell,W} = \{g_1,\dots,g_x\}$ with $g_y = (w_{y,1},\dots,w_{y,k})$, where each vertex $w_{y,h} \in W$ is chosen independently and uniformly at random. For $G \in \{F, H^*_\ell\}$ we henceforth also write $C_w(G)$ for the component of G which contains w.

Our analysis of the components of H^*_ℓ is based on a natural 'breadth first search' exploration process: in each step $j \geqslant 0$ we maintain two lists, one of 'active' vertices $\mathcal{A}_j = V(F)$, and one of 'explored' vertices $\mathcal{E}_j \subseteq V(F)$. Initially $\mathcal{A}_0 := C_{v_0}(F)$ and $\mathcal{E}_0 := \emptyset$, where $v_0 \in V(F)$ is chosen uniformly at random. In step $j \geqslant 1$, we pick an active vertex $v_j \in \mathcal{A}_{j-1} \cap W$ (if there is one) and proceed as follows. We sequentially test the presence and multiplicity of each (so far untested) tuple $g \in W^k$ of the form $(v_j, w_1, \dots, w_{k-1}), \dots, (w_1,\dots,w_{k-1},v_j)$ in $\mathcal{E}^*_{\ell,W}$, and denote the resulting multiset of newly found 'partial' tuples $\tilde{g} = (w_1,\dots,w_{k-1})$ by $\mathfrak{S}_j$. For each $\tilde{g} \in \mathfrak{S}_j$, we then mark all vertices in $\bigcup_{1\leqslant h<k} C_{w_h}(F) \setminus (\mathcal{A}_{j-1} \cup \mathcal{E}_{j-1})$ as active. At the end of step j we move v_j from the active list to the explored list. The exploration process stops when $|\mathcal{A}_j \cap W| = 0$, in which case $\mathcal{E}_j \cup \mathcal{A}_j = C_{v_0}(H^*_\ell)$ holds (since we already found all hyperedges of $\mathcal{E}^*_{\ell,W}$ containing a vertex from $\mathcal{E}_j$).

By construction $\mathbb{P}(|C_{v_0}(H^*_\ell)| \geqslant s \mid H^*_\ell = G) = N_{\geqslant s}(G)/|V(F)|$, so using $|V(F)| = n$ we see that

$$\mathbb{E}N_{\geqslant s}(H^*_\ell) = \mathbb{P}(|C_{v_0}(H^*_\ell)| \geqslant s)n. \tag{C.1}$$

eq:Ngej:UB

Formally setting $\mathcal{A}_{j+1} := \mathcal{A}_j$ and $\mathcal{E}_{j+1} := \mathcal{E}_j$ whenever $|\mathcal{A}_j \cap W| = 0$, let

$$X_j := |(\mathcal{A}_j \cup \mathcal{E}_j) \cap W| \quad \text{and} \quad Y_j := |\mathcal{A}_j \cup \mathcal{E}_j|. \tag{C.2}$$

def:XjYj

At the end of each step $v_j \in \mathcal{A}_{j-1} \cap W$ is moved from the active to the explored list, so $X_r \leqslant r$ implies $|\mathcal{A}_r \cap W| = 0$ and thus $Y_r = |C_{v_0}(H^*_\ell)|$. For all $r \geqslant 0$, it follows that

$$\mathbb{P}(|C_{v_0}(H^*_\ell)| \geqslant s) \leqslant \mathbb{P}(X_r > r) + \mathbb{P}(Y_r \geqslant s). \tag{C.3}$$

eq:Pr:J

We next show that the component size distribution of H^*_ℓ has an exponential tail, by estimating X_r and Y_r using the following Chernoff-type bound (whose standard proof we include for completeness).

:sum:T

Lemma C.1. *Let $Z, Z_0 \geqslant 0$ be independent integer-valued random variables with $\mathbb{E}Z \leqslant \mu - \gamma$, $\mathbb{E}\alpha^{Z_0} \leqslant A$ and $\mathbb{E}\alpha^{Z} \leqslant A'$, where $\alpha > 1$. Given $r \geqslant 0$ set $T_r := Z_0 + \sum_{1\leqslant j\leqslant r} Z_j$, where the Z_j with $j \geqslant 1$ are independent copies of Z. Then there is $a = a(\mu,\gamma,\alpha,A') > 0$ such that $\mathbb{P}(T_r \geqslant s) \leqslant Ae^{-as}$ for all $s \geqslant \mu r$.*

Proof. Let $f(t) := \mathbb{E}(e^{t(Z-\mu)})$. Clearly, $f(0) = 1$ and $f'(0) = \mathbb{E}(Z-\mu) \leq -\gamma$. Since $\mathbb{P}(Z \geqslant s)\alpha^s \leqslant \mathbb{E}\alpha^Z \leqslant A$, there is $D = D(\mu,\alpha,A) > 0$ such that $f''(t) = \mathbb{E}((Z-\mu)^2 e^{t(Z-\mu)}) \leq D$ for $0 \leq t \leq (\log\alpha)/2$, say. For $x := \min\{\gamma/D, (\log\alpha)/2\}$ Taylor's Theorem thus yields $f(x) \leq 1 - \gamma x + Dx^2/2 \leq 1 - \gamma x/2 =: c$, with $c < 1$. Recalling $s \geqslant \mu r$, using Markov's inequality and independence of the Z_j we infer

$$\mathbb{P}(T_r \geqslant s) = \mathbb{P}\big(e^{x(T_r-\mu r)} \geqslant e^{x(s-\mu r)}\big) \leqslant \big(\mathbb{E}e^{xZ_0}\big)\big(f(x)\big)^r \cdot e^{-x(s-\mu r)} \leqslant Ac^r e^{-x(s-\mu r)} \leqslant Ae^{-as}$$

for suitable $a = a(c,\mu,x) > 0$ (by distinguishing the cases $s \geqslant 2\mu r$ and $r \geqslant s/(2\mu)$, say). □

:UT:XY

Lemma C.2. *If the assumptions of the subcritical case of Lemma 2.3 hold, then there are constants $c, C > 0$ (depending only on k, β, B, ξ, γ) such that $\mathbb{P}(|C_{v_0}(H^*_\ell)| \geqslant s) \leqslant Ce^{-cs}$ for all $s \geqslant 0$.*

Proof. Starting with upper tail bounds for X_r, note that we initially have

$$X_0, \leqslant |C_{v_0}(F)| =: N_0. \tag{C.4}$$ eq:deltaX0

In each subsequent exploration step $j \geqslant 1$, the multiset of newly 'found' partial tuples $\{(w_1, \dots, w_{k-1}) \in \mathfrak{S}_j\}$ is dominated from above (with respect to the subset relation) by the random multiset $\mathfrak{S}$, where tuples $(w_1, \dots, w_{k-1}) \in W^{k-1}$ appear according to independent Poisson processes with rate $k\psi = k\ell/|W|^k$. It follows that there is a coupling of X_r and T_r satisfying

$$X_r \leqslant N_0 + \sum_{1 \leqslant j \leqslant r} R_j =: T_r, \tag{C.5}$$ eq:deltaXj:cpl

where the R_j are independent copies of

$$R := \sum_{(w_1,\dots,w_{k-1}) \in \mathfrak{S}} \sum_{1 \leqslant h < k} |C_{w_h}(F) \cap W|. \tag{C.6}$$ eq:deltaXj:1

Similarly to $\mathcal{E}^*_{\ell,W}$ we may generate $\mathfrak{S}$ as follows: we first determine $x := |\mathfrak{S}| \sim \mathrm{Po}(k\psi|W|^{k-1})$, and then set $\mathfrak{S} = \{g_1, \dots, g_x\}$ with $g_y = (w_{y,1}, \dots, w_{y,k-1})$, where each vertex $w_{y,h} \in W$ is chosen independently and uniformly at random. Let $N \sim |C_w(F) \cap W|$, where $w \in W$ is chosen uniformly at random. Using $\mathbb{E}N = S(F, W)$ and $\mathbb{E}|\mathfrak{S}| = k\psi|W|^{k-1} = k\ell/|W|$ together with the subcritical condition, it follows that

$$\mathbb{E}R = \mathbb{E}|\mathfrak{S}| \cdot (k-1) \cdot \mathbb{E}N = k\ell/|W| \cdot (k-1) \cdot S(F, W) \leqslant 1 - \gamma. \tag{C.7}$$ eq:Rj:Exp

Let $\alpha := \beta^{1/2}$. Recalling (C.6), using the two-stage construction of $\mathfrak{S}$ it is routine to deduce

$$\mathbb{E}\alpha^R = \mathbb{E}\Big(\Big[\big(\mathbb{E}\alpha^N\big)^{k-1} \Big]^{|\mathfrak{S}|} \Big) \leqslant \exp\Big(\mathbb{E}|\mathfrak{S}| \cdot \big(\mathbb{E}\alpha^N\big)^{k-1} \Big).$$

Noting $(k-1)\mathbb{E}N \geqslant 1$, we see that (C.7) yields $\mathbb{E}|\mathfrak{S}| \leqslant 1 - \gamma$. Using the main technical assumptions we infer $\mathbb{P}(N \geqslant s) \leqslant N_{\geqslant s}(F)/|W| \leqslant B\beta^{-s}/\xi$, which in view of $\beta = \alpha^2$ implies $\mathbb{E}\alpha^N \leqslant B/[\xi(1-\alpha^{-1})]$. Hence $\mathbb{E}\alpha^R \leqslant A'$ for suitable $A' = A'(k, B, \xi, \alpha)$. Since $\mathbb{P}(N_0 \geqslant s) \leqslant N_{\geqslant s}(F)/n$, we analogously infer $\mathbb{E}\alpha^{N_0} \leqslant B/(1-\alpha^{-1}) =: A$. Invoking Lemma C.1 with $\mu = 1$, there is $a = a(\gamma, \alpha, A') > 0$ such that

$$\mathbb{P}(X_r > r) \leqslant \mathbb{P}(T_r \geqslant r) \leqslant Ae^{-ar} \quad \text{for all } r \geqslant 0. \tag{C.8}$$ eq:Xr:tail

Mimicking the above analysis for Y_r, it follows that there is a coupling of Y_r and T^+_r satisfying

$$Y_r \leqslant N_0 + \sum_{1 \leqslant j \leqslant r} R^+_j =: T^+_r, \tag{C.9}$$ eq:deltaYj:cpl

where the R^+_j are independent copies of the random variable R^+, which is defined analogously to R but with $|C_{w_h}(F) \cap W|$ is replaced by $|C_{w_h}(F)|$ in (C.6). Let $N^+ \sim |C_w(F)|$, where $w \in W$ is chosen uniformly at random. Similarly to N above, we here again have $\mathbb{P}(N^+ \geqslant s) \leqslant N_{\geqslant s}(F)/|W| \leqslant B\beta^{-s}/\xi$, which in turn implies $\mathbb{E}\alpha^{N^+} \leqslant B/[\xi(1-\alpha^{-1})]$ and $\mathbb{E}\alpha^{R'} \leqslant A'$, as well as $\mathbb{E}N^+ = \sum_{s \geqslant 0} \mathbb{P}(N^+ \geqslant s) \leqslant B/[\xi(1-\beta^{-1})] =: \lambda$. Analogously to (C.7) we also have $\mathbb{E}|R^+| = \mathbb{E}|\mathfrak{S}| \cdot (k-1) \cdot \mathbb{E}N^+ \leqslant (k-1)\lambda$. Invoking Lemma C.1 with $\mu = \lambda k$ and $\gamma = \lambda$, similarly to (C.8) there is $b = b(\lambda, k, \alpha, A') > 0$ such that

$$\mathbb{P}(Y_r \geqslant s) \leqslant \mathbb{P}(T^+_r \geqslant s) \leqslant Ae^{-bs} \quad \text{for all } s \geqslant \lambda k r \text{ and } r \geqslant 0. \tag{C.10}$$ eq:Yr:tail

Finally, set $r := \lfloor s/(\lambda k) \rfloor$. If $s \geqslant 2\lambda k$, then (C.8) and (C.10) imply $\mathbb{P}(|C_{v_0}(H^*_\ell)| \geqslant s) \leqslant A(e^{-ar} + e^{-bs})$, with $r \geqslant s/(2\lambda k)$. Since otherwise trivially $\mathbb{P}(|C_{v_0}(H^*_\ell)| \geqslant s) \leqslant 1$, this completes the proof for suitable constants $c, C > 0$ (for example, $c := \min\{a/(2\lambda k), b\}$ and $C := \max\{2A, e^{2\lambda k c}\}$ suffice). □

The following proof uses a truncation argument, which restricts to components of size $O(\log n)$.

Proof of the 'subcritical case' of Lemma 2.3. For $c, C > 0$ as given by Lemma C.2, set $D := (2+\pi)/c$ and $s := D \log n$. Applying Markov's inequality, using (C.1) and Lemma C.2 we obtain that

$$\mathbb{P}(L_1(H_\ell^*) \geqslant s) \leqslant \mathbb{P}(N_{\geqslant s}(H_\ell^*) \geqslant s) \leqslant \mathbb{E}N_{\geqslant s}(H_\ell^*) = \mathbb{P}(|C_{v_0}(H_\ell^*)| \geqslant s)n \leqslant Cn^{-(1+\pi)}.$$

Since $H_\ell^* = F + \mathcal{E}_{\ell,W}^*$ conditioned on $|\mathcal{E}_{\ell,W}^*| = \ell$ has the same distribution as $H_\ell = F + \mathcal{E}_{\ell,W}$, using $|\mathcal{E}_{\ell,W}^*| \sim \mathrm{Po}(\ell)$ and $\ell \leqslant (1-\gamma)|W|/[k(k-1)S(F,W)] \leqslant |W| \leqslant n$ (by the subcritical condition) we infer

$$\mathbb{P}(L_1(H_\ell) \geqslant s) = \mathbb{P}\big(L_1(H_\ell^*) \geqslant s \,\big|\, |\mathcal{E}_{\ell,W}^*| = \ell\big) \leqslant O\big(\sqrt{\ell}\big) \cdot \mathbb{P}(L_1(H_\ell^*) \geqslant s) \leqslant o(n^{-\pi}). \tag{C.11}$$

eq:pittel:L1

Furthermore, since $N_{\geqslant j}(\cdot)$ is kj-Lipschitz with respect to the addition or deletion of hyperedges, using $|\mathcal{E}_{\ell,W}^*| \sim \mathrm{Po}(\ell)$ and Jensen's inequality together with $\ell \leqslant n$ it follows that

$$|\mathbb{E}N_{\geqslant j}(H_\ell^*) - \mathbb{E}N_{\geqslant j}(H_\ell)| \leqslant kj \cdot \mathbb{E}\big||\mathcal{E}_{\ell,W}^*| - \ell\big| \leqslant kj \cdot \sqrt{\mathrm{Var}\,|\mathcal{E}_{\ell,W}^*|} \leqslant kj\sqrt{n}. \tag{C.12}$$

eq:ENj:approx

With an eye on (C.11), we now define $\alpha := \min\{e^{c/2}, e^{1/(5D)}\} > 1$ and

$$Z := \sum_{1 \leqslant j \leqslant s} \alpha^j N_{\geqslant j}(H_\ell). \tag{C.13}$$

def:Zi

Using (C.12) together with (C.1) and Lemma C.2, by choice of α and $s = D \log n$ it follows that

$$\mathbb{E}Z \leqslant \sum_{1 \leqslant j \leqslant s} \alpha^j \cdot \Big(\mathbb{E}N_{\geqslant j}(H_\ell^*) + kj\sqrt{n}\Big) \leqslant \sum_{j \geqslant 1} Ce^{-cj/2}n + n^{1/5} \cdot ks^2\sqrt{n} \leqslant A_0 n \tag{C.14}$$

eq:E:Zi

for suitable $A_0 = A_0(c, C, k, D)$. Observe that H_ℓ is equivalent to a probability space Ω that consists of $k\ell = O(n)$ independent random variables, each of which corresponds to the choice of a random vertex from W. Furthermore, changing the outcome of one variable can be interpreted as first deleting one hyperedge and then adding one hyperedge, which in turn changes each $N_{\geqslant j}(\cdot)$ by at most $2kj$. So, whenever $\omega_1, \omega_2 \in \Omega$ differ in the outcome of at most one random variable, then by choice of α and s it follows that

$$|Z(\omega_1) - Z(\omega_2)| \leqslant \sum_{1 \leqslant j \leqslant s} \alpha^j \cdot 2kj \leqslant 2ks^2 n^{1/5} = o(n^{1/4}).$$

Hence a standard application of the bounded differences inequality [35] yields $\mathbb{P}(Z \geqslant \mathbb{E}Z + n) \leqslant n^{-\omega(1)}$, which together with (C.11) and (C.13)–(C.14) establishes the subcritical case of Lemma 2.3 with $A := A_0 + 1$. □

C.2 Lemma 2.3: supercritical case (adding a random matching)

upercr

In this appendix we prove the supercritical case of Lemma 2.3 from Section 2.2 for $M_\ell = F + \mathcal{M}_{\ell,W}$. For technical reasons we shall generate a random k-matching of W of size ℓ using the following procedure.

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Lemma C.3. *Let $\ell, N, k \in \mathbb{N}$ satisfy $k \geqslant 1$ and $N \geqslant \max\{\ell k, 1\}$. Then, given $W \subseteq \mathbb{N}$ with $|W| = N$, the following procedure generates a uniform random k-matching $\mathcal{M}$ of W with $|\mathcal{M}| = \ell$. Starting with $m := \ell$, $\mathcal{M} := \emptyset$, $V_1 := W$ and $j := 1$, repeat the following as long as $m \geq 1$. Pick any $v_j \in V_j$, and flip an independent random coin with success probability $km/|V_j|$. In case of* success*, add the hyperedge $e := \{v_j, w_1, \ldots, w_{k-1}\}$ to $\mathcal{M}$, where each $w_i \in V_j \setminus \{v_j, w_1, \ldots, w_{i-1}\}$ is chosen independently and uniformly at random. Furthermore, set $m := m-1$, $V_{j+1} := V_j \setminus e$ and $j := j+1$. In case of* failure*, set $V_{j+1} := V_j \setminus \{v_j\}$ and $j := j+1$.*

Proof. Fix $k \geqslant 1$. It suffices to show, by induction on $\ell + N$ satisfying $N \geqslant \max\{\ell k, 1\}$, that every k-matching $\mathcal{M}^*$ of $W \subseteq \mathbb{N}$ with $|\mathcal{M}^*| = \ell$ and $|W| = N$ is generated with probability

$$p_k(\ell, N) := \mathbb{1}_{\{\ell \geqslant 1\}} \frac{\ell!}{\prod_{j=0}^{\ell-1} \binom{N-jk}{k}} + \mathbb{1}_{\{\ell = 0\}}.$$

For the base case, note that the claim is trivial whenever $\ell = 0$ or $1 \leqslant N \leqslant k$. For the induction step we thus may assume that $\ell + N$ satisfies $\ell \geqslant 1$ and $N \geqslant \max\{\ell k, k+1\}$. Fix an arbitrary k-matching $\mathcal{M}^*$ of W

with $|\mathcal{M}^*| = \ell$. Let $v_j \in W$ be the first vertex picked by the procedure, for which we distinguish two cases. First, if no hyperedge of $\mathcal{M}^*$ contains v_j, then we must have $N-1 \geqslant \ell k$. Since also $N-1 \geqslant k \geqslant 1$ and $\ell \geqslant 1$, using induction it then follows that the procedure generates $\mathcal{M}^*$ with probability

$$\left(1 - \frac{k\ell}{N}\right) \cdot p_k(\ell, N-1) = \frac{N-k\ell}{N} \cdot \frac{\ell!}{\prod_{j=0}^{\ell-1} \binom{N-1-jk}{k}} = \frac{\ell!}{\prod_{j=0}^{\ell-1} \binom{N-jk}{k}} = p_k(\ell, N).$$

Second, if v_j is contained in some hyperedge of $\mathcal{M}^*$, then the procedure adds the corresponding unique hyperedge $f = \{v_j, w_1, \dots, w_{k-1}\}$ with probability

$$q_f := \frac{k\ell}{N} \cdot \left(\mathbb{1}_{\{k \geqslant 2\}} \frac{(k-1)!}{\prod_{i=1}^{k-1}(N-i)} + \mathbb{1}_{\{k=1\}} \right) = \frac{\ell}{\binom{N}{k}}.$$

Since $N - k \geqslant \max\{(\ell-1)k, 1\}$, using induction it follows that the procedure generates $\mathcal{M}^*$ with probability

$$q_f \cdot p_k(\ell-1, N-k) = \frac{\ell}{\binom{N}{k}} \cdot \left(\mathbb{1}_{\{\ell \geqslant 2\}} \frac{(\ell-1)!}{\prod_{j=0}^{\ell-2} \binom{N-k-jk}{k}} + \mathbb{1}_{\{\ell=1\}} \right) = p_k(\ell, N),$$

completing the proof. □

Deferring the choice of $K \geqslant 1$, it will be convenient to study the 'truncated' subgraph $F_K \subseteq F$, where we ensure (by deleting suitable edges) that all vertices in components C of F with $|C \cap W| > K$ become isolated vertices in F_K. In particular, any component C of F_K satisfies $|C \cap W| \leqslant K$. Furthermore,

$$M_{\ell,K} := F_K + \mathcal{M}_{\ell,W} \subseteq F + \mathcal{M}_{\ell,W} = M_\ell. \tag{C.15}$$ def:Fk

Recalling the definition of $S(F,W)$ from Lemma 2.3, by construction of F_K we also have

$$\big|S(F,W) - S(F_K,W)\big| \leqslant \sum_{\substack{w \in W: \\ |C_w(F) \cap W| > K}} |C_w(F) \cap W|/|W| \leqslant \sum_{s>K} s N_{\geqslant s}(F)/|W|, \tag{C.16}$$ eq:SFKW:lower

where for $G \in \{F, F_K\}$ we write $C_w(G)$ for the component of G which contains w, as before.

Our analysis of of $M_{\ell,K}$ is again based on an exploration process, but with the twist that after finishing exploring one component we start exploring another component: in each step $j \geqslant 0$ we maintain lists of 'active' vertices $\mathcal{A}_j \subseteq W$ and 'explored' vertices $\mathcal{E}_j \subseteq W$. Using the procedure from Lemma C.3, the plan is to step-by-step generate the matching $\mathcal{M}_{\ell,W}$ as the exploration process evolves, formally also maintaining the list of so-far 'unexplored' vertices $V_j := W \setminus \mathcal{E}_{j-1}$, and the 'remaining' number of matching hyperedges m. Turning to the details, initially we pick a vertex $v_0 \in W$ and start with $\mathcal{A}_0 := C_{v_0}(F_K) \cap W$, $\mathcal{E}_0 := \emptyset$ and $m := \ell$. In step $j \geqslant 1$, when $m \geqslant 1$ holds we then pick an active vertex $v_j \in \mathcal{A}_{j-1} \subseteq W \setminus \mathcal{E}_{j-1} = V_j$, and intuitively test for the presence of a matching hyperedge containing v_j, which in the procedure from Lemma C.3 formally corresponds to flipping an independent random coin with success probability $km/|V_j|$. Only in case of success we proceed as follows: we generate the hyperedge $e = \{v_j, w_1, \dots, w_{k-1}\} \subseteq V_j$ as in Lemma C.3, then mark the vertices in $e^* := \{w_1, \dots, w_{k-1}\}$ as explored, mark all vertices in $\bigcup_{1 \leqslant h < k} (C_{w_h}(F_K) \cap W) \setminus (\mathcal{A}_{j-1} \cup \mathcal{E}_{j-1} \cup e^*)$ as active, and set $m := m - 1$. At the end of step j we move v_j from the active list to the explored list. If the resulting active list $\mathcal{A}_j$ is empty, then we pick an unexplored vertex $v_j^* \in W \setminus \mathcal{E}_j$ (if there is one), and set $\mathcal{A}_j := C_{v_j^*}(F_K) \cap W$; the exploration process stops if no such v_j^* exists, or if $m = 0$.

The key observation is that, at the end of each step $j \geqslant 0$, all currently active vertices in $\mathcal{A}_j$ belong to the same component of $M_{\ell,K}$, so that (C.15) implies

$$L_1(M_\ell) \geqslant L_1(M_{\ell,K}) \geqslant |\mathcal{A}_j|. \tag{C.17}$$ eq:L1Mo

The following proof analyzes the (expected) one-step changes of the active vertex set $\mathcal{A}_j$.

Proof of 'supercritical case' of Lemma 2.3. We start by carefully choosing several constants, along with some preliminary estimates. First, we pick $\sigma > 0$ small enough such that

$$(1-\sigma)^2(1+\gamma) - 1 \geqslant \gamma/2. \tag{C.18}$$ choice:sigma

Second, using the supercritical condition and $\ell \leqslant |W|/k$ it follows that

$$S(F,W) - 1 \geqslant \frac{(1+\gamma)|W|}{k(k-1)\ell} > \frac{1}{k}. \tag{C.19}$$ ineq:S

We now pick $K = K(k,\beta,B,\xi,\gamma) \geqslant 1$ large enough such that the main technical assumptions imply

$$\sum_{s>K} sN_{\geqslant s}(F)/|W| \leqslant \sum_{s>K} sB\beta^{-s}/\xi \leqslant \frac{\sigma}{2k}. \tag{C.20}$$ choice:K

Third, as in the proof of Lemma C.2 we have $S(F,W) = \mathbb{E}N \leqslant \mathbb{E}N^+ \leqslant B/[\xi(1-\beta^{-1})]$, so using the supercritical condition it follows that that there is $\tau = \tau(k,\beta,B,\xi) > 0$ such that

$$\ell \geqslant \frac{(1+\gamma)|W|}{k(k-1)\big(S(F,W)-1\big)} \geqslant \tau|W|. \tag{C.21}$$ eq:ell:upper

We now define $I := \lfloor \delta|W| \rfloor$, where we pick $\delta > 0$ small enough such that

$$\delta < \min\{\tau, \sigma\tau, 1\} \quad \text{and} \quad 5kK^2\delta \leqslant \frac{\sigma}{2k}. \tag{C.22}$$ choice:delta

We henceforth restrict our attention to the first I steps of the exploration process, during which $m \geqslant \ell - I \geqslant (\tau-\delta)|W| > 0$ and $|W \setminus \mathcal{E}_j| \geqslant |W| - I \geqslant (1-\delta)|W| > 0$ ensure that the process never stops. As usual, we denote by $\mathcal{F}_j$ the natural filtration associated to the exploration process after j steps. Writing $v_j \in \mathcal{A}_{j-1} \subseteq W \setminus \mathcal{E}_{j-1} = V_j$ for the active vertex chosen in step $1 \leqslant j \leqslant I$, it is not difficult to see that

$$\mathbb{E}\big(|\mathcal{A}_j| - |\mathcal{A}_{j-1}| \,\big|\, \mathcal{F}_j\big) \geqslant \frac{k(\ell-I)}{|W|} \cdot \mathbb{E}\Big(\sum_{1\leqslant h<k} \mathbb{E}\big(|\mathfrak{C}_{w_h}| \,\big|\, \mathcal{F}_{j-1}, w_1, \dots, w_{h-1}\big) \,\Big|\, \mathcal{F}_{j-1}\Big) - 1, \tag{C.23}$$ eq:evo:EAj

where the vertices $w_h \in W_h := W \setminus (\mathcal{E}_{j-1} \cup \{v_j, w_1, \dots, w_{h-1}\}) = V_j \setminus \{v_j, w_1, \dots, w_{h-1}\}$ are chosen independently and uniformly, and the vertex set $\mathfrak{C}_{w_h}$ is defined as

$$\mathfrak{C}_{w_h} := \big(C_{w_h}(F_K) \cap W\big) \setminus \Big(\mathcal{A}_{j-1} \cup \mathcal{E}_{j-1} \cup \{w_1, \dots, w_{k-1}\} \cup \bigcup_{1\leqslant s<k : s\neq h} \big(C_{w_s}(F_K) \cap W\big)\Big).$$

Using $w_h \in C_{w_h}(F_K) \cap W$ and $|C_w(F_K) \cap W| \leqslant K$ together with $|\mathcal{A}_{j-1} \cup \mathcal{E}_{j-1}| \leqslant IkK$, a moment's thought reveals that we *deterministically* (over all possible choices of the vertices w_s with $s \neq h$) have

$$\begin{aligned}\sum_{w\in W_h} |\mathfrak{C}_w| &\geqslant \sum_{w\in W_h} \big|(C_w(F_K)\cap W) \setminus \{w\}\big| - \big(|\mathcal{A}_{j-1} \cup \mathcal{E}_{j-1}| + k + kK\big) \cdot K \\ &\geqslant \sum_{w\in W} \big(|C_w(F_K)\cap W| - 1\big) - 5kK^2 I.\end{aligned}$$

Noting $|W_h| \leqslant |W|$, by combining $I \leqslant \delta|W|$ and (C.22) with (C.16) and (C.20) it follows that

$$\sum_{w\in W_h} \frac{|\mathfrak{C}_w|}{|W_h|} \geqslant S(F_K,W) - 1 - 5kK^2\delta \geqslant S(F,W) - 1 - \frac{\sigma}{k} \geqslant (1-\sigma)\cdot\big(S(F,W)-1\big),$$

where we used (C.19) for the last inequality. Note that (C.21)–(C.22) imply $I \leqslant \delta|W| \leqslant \sigma\ell$. Using the supercritical condition together with our choice (C.18) of σ, it follows that (C.23) is at least

$$\begin{aligned}\mathbb{E}\big(|\mathcal{A}_j| - |\mathcal{A}_{j-1}| \,\big|\, \mathcal{F}_j\big) &\geqslant \frac{k(1-\sigma)\ell}{|W|} \cdot (k-1) \cdot (1-\sigma)\big(S(F,W)-1\big) - 1 \\ &\geqslant (1-\sigma)^2 \cdot (1+\gamma) - 1 \geqslant \gamma/2.\end{aligned} \tag{C.24}$$ eq:evo:EAj:2

Finally, set $\Delta_j := |\mathcal{A}_j| - |\mathcal{A}_{j-1}|$ and $Z_s := \sum_{1\leqslant j\leqslant s}\big[\Delta_j - \mathbb{E}(\Delta_j|\mathcal{F}_{j-1})\big]$. The expected one-step changes of $(Z_j)_{0\leqslant j\leqslant I}$ are $\mathbb{E}(Z_j - Z_{j-1} \mid \mathcal{F}_{j-1}) = 0$. Combining (C.24) and (C.17) with $|\mathcal{A}_0| \geqslant 0$, it follows that

$$Z_I = \sum_{1\leqslant j\leqslant I} \Delta_j - \sum_{1\leqslant j\leqslant I} \mathbb{E}(\Delta_j|\mathcal{F}_{j-1}) \leqslant \big(|\mathcal{A}_I| - |\mathcal{A}_0|\big) - I\cdot\gamma/2 \leqslant L_1(M_\ell) - \gamma I/2.$$

We have $\big||\mathcal{A}_j| - |\mathcal{A}_{j-1}|\big| \leqslant kK$ by definition of F_K, so the maximum one-step changes of $(Z_j)_{0\leqslant j\leqslant I}$ satisfy $|Z_j - Z_{j-1}| \leqslant 2kK$. Since $Z_0 = 0$, a standard application of the Azuma–Hoeffding inequality [6, 27] yields

$$\mathbb{P}(L_1(M_\ell) \leqslant \gamma I/4) \leqslant \mathbb{P}(Z_I \leqslant -\gamma I/4) \leqslant e^{-I/[2(2kK)^2]},$$

which together with $I \geqslant \delta|W|/2 \geqslant \delta\xi n/2$ (for n large enough) establishes the supercritical case of Lemma 2.3 for suitable $c, \lambda > 0$. □

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C.3 Lemma 4.3: largest component of random 2-regular graph

Proof of Lemma 4.3. Let X_t denote the number of cycles of length t in R_m. Abbreviating the number of perfect matchings of $2i$ points by $M(2i) := (2i)!/(i!2^i)$, simple computations (see also [24, Lemma 4]) yield

$$\mathbb{E}X_t = \binom{m}{t} \cdot 2^t \cdot \frac{(t-1)!}{2} \cdot \frac{M(2m-2t)}{M(2m)} = \frac{([m]_t)^2 2^{2t-1}}{t[2m]_{2t}}, \tag{C.25}$$ exlong0

where $[a]_b$ denotes the falling factorial $a(a-1)\cdots(a-b+1)$. Using Stirling's formula we obtain

$$\mathbb{E}X_t = \theta \frac{\sqrt{m}}{2t\sqrt{\max\{m-t,1\}}}, \tag{C.26}$$ exlong

where $\theta \sim 1$ if $m - t \to \infty$ and $\theta = \Theta(1)$ always (i.e., when $3 \leqslant t \leqslant m$). If $m/2 < t \leqslant m$, then $X_t \in \{0,1\}$, so that $\mathbb{P}(L_1(R_m) = t) = \mathbb{E}X_t$. Summing (C.26), it is routine to verify that in (4.9) we can set

$$F(c) := \int_c^1 \frac{dx}{2x\sqrt{1-x}} = \log\big(\sqrt{c^{-1}} + \sqrt{c^{-1}-1}\big) \qquad \text{when } 1 \geqslant c \geqslant 1/2. \tag{C.27}$$ def:LF:0

If $m/3 < t \leqslant m/2$, then $X_t \in \{0,1,2\}$, so cycles of length t may no longer be unique. Note that the probability that the longest cycle of R_m has length t and is unique, is precisely the expected number of cycles of length t for which the remaining graph on $m - t$ vertices has longest cycle length at most $t - 1$. By the conclusion of (C.25)–(C.27) above, the probability that the remaining graph has longest cycle length greater than t is asymptotic to $F(t/(m-t))$. Furthermore, the probability that R_m has two cycles of length t is routinely seen to be (by simple calculations similar to (C.25)–(C.26) above) at most

$$\binom{m}{t}\binom{m-t}{t} \cdot \left(2^t \cdot \frac{(t-1)!}{2}\right)^2 \cdot \frac{M(2m-4t)}{M(2m)} = \frac{O(\sqrt{m})}{t^2\sqrt{\max\{m-2t,1\}}} = o(m^{-1}).$$

Putting things together, it follows similarly to (C.27) that in (4.9) we can set

$$F(c) := F(1/2) + \int_c^{1/2} \frac{1 - F\big(x/(1-x)\big)}{2x\sqrt{1-x}}\, dx \qquad \text{when } 1/2 > c \geqslant 1/3,$$

where the recursive call of F is well-defined since $x/(1-x) \in [1/2, 1]$. Iterating the same reasoning, it follows that in (4.9) we can more generally set

$$F(c) := F(1/k) + \int_c^{1/k} \frac{1 - F\big(x/(1-x)\big)}{2x\sqrt{1-x}}\, dx \qquad \text{when } 1/k > c \geqslant 1/(k+1) \text{ for } k \geqslant 2. \tag{C.28}$$ def:LF:k

Turning to the claimed properties of the function F, note that (4.9) implies $F(c) \in [0,1]$ for all $c \in (0,1]$. Using (C.27) and (C.28), it is routine to check that F is monotone decreasing and continuous on $(0,1]$, and continuously differentiable on $(0,1)$. In addition, for any $c \in (0,1)$ we have

$$1 \geqslant F(c) \geqslant \big(1 - \max_{y\in[c,1]} F(y)\big) \cdot \int_c^1 \frac{1}{2x\sqrt{1-x}}\, dx \geqslant \big(1 - F(c)\big) \cdot \log\big(\sqrt{c^{-1}} + \sqrt{c^{-1}-1}\big),$$

so that $F(c) \geqslant 1 - 1/\log(\sqrt{c^{-1}}) \to 1$ as $c \to 0$. We now show that $F(c) \in (0,1)$ for all $c \in (0,1)$. The lower bound is immediate, since $F(c) \geqslant F\big(\max\{c, 1/2\}\big) > 0$ by (C.27). For the upper bound, we

similarly see that $F(c) \leqslant F(1/2) < 1$ for any $c \in [1/2, 1]$. For $c \in (0, 1/2)$ there are several ways to prove that $F(c) < 1$. One (somewhat formal) proof proceeds by contradiction, assuming that $F(c) = 1$ for some $1/(k+2) \leqslant c < 1/(k+1)$ with $k \geqslant 1$. Then monotonicity gives $F(x) = 1$ on $[0, c]$, which in turn implies $F'(c) = 0$, so that $F(c/(1-c)) = 1$ follows from (C.28). Iterating this reasoning, it follows that $F(c_k) = 1$ for $c_k = c/(1-kc) \in [1/2, 1)$, contradicting that $F(c_k) \leqslant F(1/2) < 1$. A second (perhaps more insightful) proof exploits that (4.9) implies $1 - F(1/K) = \mathbb{P}(L_1(R_m) \leqslant m/K) + o(1)$ for any $K \geqslant 2$. Note that the number Y_K of sets of K distinct cycles with lengths all between $1 + m/(K+1)$ and m/K satisfies $Y_K \in \{0, 1\}$, so that $\mathbb{P}(Y_K = 1) = \mathbb{E}Y_K$. Furthermore, $\mathbb{E}Y_K = \mu_K + o(1)$ where μ_K is the value of an iterated integral similar to (C.28), and clearly $\mu_K > 0$. Since $Y_K = 1$ implies $L_1(R_m) \leqslant m/K$, it follows that $F(c) \leqslant F(1/K) \leqslant 1 - \mu_K < 1$ for all $c \in [1/K, 1]$, completing the proof since $K \geqslant 2$ was arbitrary. $\square$