A dynamical system related to GIT

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A gradient system

- Let $\phi \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial that is homogeneous of degree m such that $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$. We consider the gradient system

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$$\frac{dx}{dt} = -\nabla\phi(x)$$
- Note that
$$\langle \nabla\phi(x), x \rangle = m\phi(x)$$
Denoting by $F(t, x)$ the solution to the system near $t = 0$ with $F(0, x) = x$. Then
$$\frac{d}{dt} \langle F(t, x), F(t, x) \rangle = -2 \langle \nabla\phi(F(t, x)), F(t, x) \rangle$$
$$= -2m\phi(F(t, x)) \leq 0.$$

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combined with the Schwarz inequality implies that
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- The Lojasiewicz gradient inequality implies the following improvement. There exists $0 < \varepsilon \leq \frac{1}{m-1}$ and $C > 0$ both depending only on ϕ such that
$$\|\nabla\phi(x)\|^{1+\varepsilon} \|x\|^{1-(m-1)\varepsilon} \geq C\phi(x).$$

- We take ε and C as above (but allow $\varepsilon = 0$ which is easy). If we write F for $F(t, X)$ and $H(t) = \phi(F(t, x))$ then we have

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- If $t \geq 0$ and $\|x\| \leq r$

$$\left\|\nabla\phi(F)\right\|^{1+\varepsilon} r^{1-(m-1)\varepsilon} \geq \left\|\nabla\phi(F)\right\|^{1+\varepsilon} \left\|F\right\|^{1-(m-1)\varepsilon} \geq C\phi(x).$$

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- We will now run through what has come to be called "the Lojasiewicz argument" which I learned from a beautiful exposition of Neeman's theorem by Gerry Schwarz.

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- $$|H'(t)| \geq \frac{1}{2}\left(\frac{C}{r^{1-3\varepsilon}}\right)^{\frac{2}{1+\varepsilon}}\phi(F)^{\frac{2}{1+\varepsilon}} = C_1(r)H(t)^{\frac{2}{1+\varepsilon}}.$$

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- $$\frac{d}{dt}H(t)^{-\frac{1-\varepsilon}{1+\varepsilon}} = -\frac{1-\varepsilon}{1+\varepsilon}\frac{H'(t)}{H(t)^{\frac{2}{1+\varepsilon}}} \geq C_1(r)$$

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- This is the first half of the calculus part of the Lojasiewicz argument. The first implication needs only the easy case $\varepsilon = 0$. If $\|x\| \leq r$ then

  $$\phi(F(t,x)) \leq \frac{C(r)}{t}$$

  so $\lim_{t\to+\infty} \phi(F(t,x)) = 0$ uniformly for x in compacta. We now do the rest of the Lojasiewicz argument which uses the existence of $\varepsilon > 0$.

- Let $f(t) = t^{1+\delta}$ with $0 < \delta < \varepsilon$ then for $t > 0$

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- $$H(s)f(s) - H(t)f(t) = \int_t^s \frac{d}{du}(H(u)f(u))du =$$

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$$\lim_{s\to+\infty} \int_t^s \left|H'(u)\right| f(u)du = \int_t^\infty H(u)f'(u)du + H(t)f(t).$$

- Thus $\sqrt{|H'(u)|\, f(u)}$ is in $L^2([t,+\infty))$ for all $t > 0$ and so
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converges absolutely and uniformly for $\|x\| \le r$.
- Noting that if $s>t$ then
$$\int_t^{s} \frac{d}{du}F(u,x)du = F(s,x) - F(t,x)$$
we have for $t>0$
$$\lim_{s\to\infty} F(s,x) = \int_t^{\infty} \frac{d}{du}F(u,x)du + F(t,x).$$

- Finally, set $L(t,x) = F(\frac{t}{1-t}, x)$ and define $L(1,x)$ by the limit above then $L : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and since

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- **Corollary.** If $Z \subset \mathbb{R}^n$ is closed and such that $F(t,z) \in Z$ for $t \geq 0$ and $z \in Z$ then $H : [0,1] \times Z \to Z$ defines a strong deformation retraction of Z onto $Z \cap Y$.

Kempf-Ness over the reals

- Let G be an open subgroup of a Zariski closed subgroup of $GL(n,\mathbb{R})$ that is closed under real adjoint relative to the standard inner product,$\langle ..., ... \rangle$, $g \to g^*$. Let $K = G \cap O(n)$. Then K is a maximal compact subgroup of G. On $\mathfrak{g} = Lie(G)$ we put the inner product $\langle X, Y \rangle = tr(XY^*)$, Set $\mathfrak{p}$ $=Lie(K)^{\perp}$ relative to this inner product.

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- We say that an element $v \in \mathbb{R}^n$ is G-critical if for any $X \in Lie(G)$, $\langle Xv, v \rangle = 0$. The following is an extension of the Kempf-Ness Theorem first observed by Richardson and Slodoway.

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3. If Gv is closed then there exists a critical element in Gv.
4. If v is critical then Gv is closed.

- We set $V = \mathbb{R}^n$ as a G–module and $Crit_G(V)$ equal to the set of all critical vectors. If $X_1, ..., X_r$ is an orthonormal basis of $\mathfrak{p}$ then

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- We consider $\mathbb{R}^n$ as $n \times 1$ columns and thus if $v \in V$ then v^* is v as a row vector. So for $v, w \in V$, vw^* is an $n \times n$ matrix and

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- Also note that $\nabla\phi(kv) = k\nabla\phi(v)$ for $k \in K$.

- Let $F(t,x)$ be the gradient flow corresponding to ϕ. Then we have shown using freshman calculus that for $t > 0$ and $\|x\| \leq r$

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- This is a very useful result since if G is connected K is connected and this implies that $Z//G$ has path lifting. In the complex case this is an important result of Kraft, Petrie and Randall.

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$$\phi(F(t,x)) \leq \frac{C(r)}{t}.$$
- In addition if $Z \subset V$ is closed and G–invariant then $F(t,Z) \subset Z$ and 2 in the real Kempf-Ness theorem implies:
- **Theorem.** Setting $L(t,Kv) = KF(\frac{t}{1-t},v)$ $0 \leq t < 1$ then $\lim_{t\to 1} L(t,Kv)$ converges uniformly on compacta and this yields a strict deformation retraction of Z/K to $(Crit_G(V) \cap Z)/K$ for any G–invariant closed subset of V.
- The statement of the next result is simplified.
- **Corollary.** If Z is Zariski closed in V then the GIT quotient, $Z//G$, of Z is a strict deformation retract of Z/K.
- This is a very useful result since if G is connected K is connected and this implies that $Z//G$ has path lifting. In the complex case this is an important result of Kraft, Petrie and Randall.
- We now consider the result implied by using the deep results of Lojasiewicz.

- The Lojasiewicz argument implies that if we set $L(t, v) = F(\frac{t}{1-t}, v)$ then $\lim_{t \to 1} H(t, v)$ converges uniformly on compacta.

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- **Theorem.** Let $Z \subset V$ be closed and G invariant then $L : [0,1] \times Z \to Z$ defines a strong, K–equivariant deformation retraction of Z onto $Z \cap Crit_G(V)$.
- Over $\mathbb{C}$ this result is due to Neeman.

- $\mathbb{C}^n = V \oplus iV$ so as a real vector space we write it as $V \oplus V = \mathbb{R}^{2n}$. The real part of the standard Hermitian inner product on $\mathbb{C}^n$ becomes the standard inner product on $\mathbb{R}^{2n}$. $M_n(\mathbb{C})$ becomes the algebra of 2×2 block $n \times n$ matrices
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- If $X \subset \mathbb{C}^n$ is Zariski closed and defined by $f_1, ..., f_k$ in $\mathbb{C}[x_1, ..., x_n]$ then it is defined by $\phi(x, y) = \sum |f_j(x + iy)|^2$ as a real variety.

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- If $G \subset GL(n, \mathbb{C})$ is a Zariski closed subgroup invariant under adjoint then G as a subgroup of $GL(2n, \mathbb{R})$ is invariant under transpose. Furthermore, if we define the critical set for the action of G on $\mathbb{C}^n$ to be
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- The original Kempf-Ness theorem is now a special case of the real Kempf-Ness theorem since Zariski closure of complex orbits is the same as the closure in the metric topology of $\mathbb{R}^{2n}$.

- The system in the abstract for my talk is just the case of $GL(n, \mathbb{C})$ acting on $M_n(\mathbb{C})$ by conjugation. Yielding the gradient system

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- Writing $F_\infty(X) = \lim_{t \to +\infty} F(t, X)$ then $F_\infty(X)$ is a normal operator with the same eigenvalues as X.

$\mathfrak{g} \subset M_n(\mathbb{C})$, $X \in \mathfrak{g}$

$X^* \in \mathfrak{g}$. $\operatorname{tr} XY^* = \langle , \rangle$.

$\theta : \mathfrak{g} \to \mathfrak{g}$ automorphism

of order $m > 0$.

$\mathfrak{h} = \mathfrak{g}^\theta$, $V = \{X \in \mathfrak{g} \mid \theta X = \varphi X\}$

~~$\varphi^m = 1$~~ $\varphi = e^{2\pi i/m}$

H = connected subgroup corresp. to $\mathfrak{h}$.

$Ad(H)$ acts on V.

$\forall\, X, Y \in V$, $[X, X^*] \in \mathfrak{h}$.

$\mathfrak{a}$ can be chosen so that $crit_G(V) = K \cdot \mathfrak{a}$.